:hree-cycle structure," for example, contains three major Fourier compo-
ents. You should also notice that when the pendulum goes over the top, its
pectrum contains a steady-state ("dc") component.

1. Dust off your program which analyzes a $y(t)$ into Fourier components. (Alternatively, you may use a Fourier analysis tool contained in your graphics program or system library.)

2. Apply your analyzer to the solution of the forced, damped pendulum for the cases where there are one-, three-, and five-cycle structures in phase space. Deduce the major frequencies contained in these structures. (Try *not* to analyze the transient behavior.)

3. Try to deduce a relation between the Fourier components, the natural frequency ω_0, and the driving frequency ω.

4. A classic signal of chaos is a broadband, although not necessarily flat, Fourier spectrum. Examine your system for parameters that give chaotic behavior and verify this statement.

4.11 EXPLORATION: PENDULUM WITH VIBRATING PIVOT

pendulum with a vibrating pivot point is an example of a *parametric reso-
ance*. It is similar to our chaotic pendulum (14.3), but with the driving force
epending on $\sin\theta$:

$$\frac{d^2\theta}{dt^2} = -\alpha\frac{d\theta}{dt} - \left(\omega_0^2 + f\cos\omega t\right)\sin\theta. \qquad (14.16)$$

ne way of understanding the physics of this equation is to go to the rest
ame of the pivot (an accelerating reference frame) where you would say that
ere is a fictitious force that effectively leads to a sinusoidal variation of g or
ω_0^2.

Analytic [L&L 69, §25-30] as well as numeric [DeJ 92, G&T 96] studies of
is system exist. A fascinating aspect of this system is that the excitation of
s modes of vibration (overtones) occurs through a series of *bifurcations*. In
ct, when the instantaneous angular velocity $d\theta/dt$ is plotted as a function of
e strength of the driving force, the bifurcation diagram in Fig. 14.8 results.
lthough the physics is very different, this behavior is manifestly similar to
e bifurcation diagram for bug populations studied in Chapter 13, *Unusual
ynamics of Nonlinear Systems*. This behavior is, apparently, the result of
ode locking and beating with the overtones.

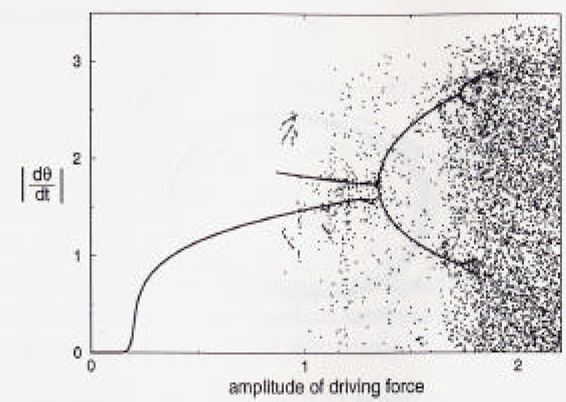

Fig. 14.8 Bifurcation diagram for the damped pendulum with a vibrating pivot. T
ordinate is $|d\theta/dt|$, the absolute value of the instantaneous angular velocity at
beginning of the period of the driver, and the abscissa is the magnitude of the driv
force d. The heavy line results from the overlapping of points, not from connect
the points. (Produced with the assistance of Melanie Johnson and Hans Kowallik.

14.11.1 Implementation: Bifurcation Diagram of Pendulum

We obtained the bifurcation diagram of Fig. 14.8 by following these steps
modification of those in [DeJ 92]):

1. Set $\alpha = 0.1$, $\omega_0 = 1$, $\omega = 2$, and let f vary through the range in Fig. 14

2. Use the initial conditions: $\theta(0) = 1$ and $\frac{d\theta}{dt}(0) = 1$.

3. Sample (record) the instantaneous angular velocity $\frac{d\theta}{dt}$ whenever t driving force passes through zero.

4. Wait 150 periods before sampling to permit transients to die off.

5. Sample $\frac{d\theta}{dt}$ for 150 times and plot the results.

6. Plot $\left|\frac{d\theta}{dt}\right|$ versus f.

7. Repeat the procedure for each new value of f.

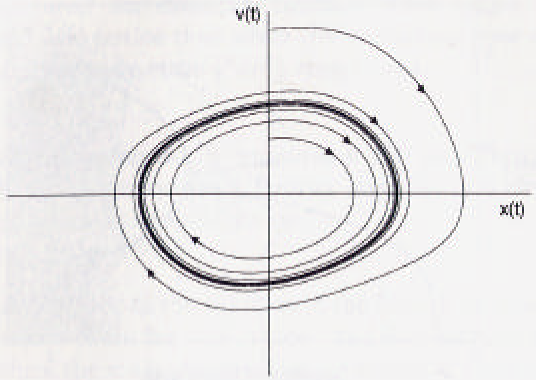

Fig. 14.9 Two phase-space trajectories corresponding to solutions of the van der Pol equation. One trajectory approaches the limit cycle (the dark curve) from the inside, while the other approaches it from the outside.

14.12 FURTHER EXPLORATIONS

1. The nonlinear behavior in once-common objects such as vacuum tubes and metronomes are described by the **van der Pool equation**

$$\frac{d^2x}{dt^2} + \mu(x^2 - x_0^2)\frac{dx}{dt} + \omega_0^2 x = 0. \qquad (14.17)$$

The behavior predicted for these systems is *self-limiting* because the equation contains a limit cycle that is also a predictable attractor. You can think of (14.17) as describing an oscillator with x-dependent damping (the μ term). If $x > x_0$, friction slows the system down; if $x < x_0$, friction speeds the system up. Some phase-space orbits are shown in Fig. 14.9. The heavy curve is the *limit cycle*. Orbits internal to the limit cycle spiral out until they reach the limit cycle; orbit external to it spiral in.

2. **Duffing oscillator:** Here we have a nonlinear oscillator that is damped and driven,

$$\frac{d^2\theta}{dt^2} - \tfrac{1}{2}\theta(1-\theta^2) = -\alpha\frac{d\theta}{dt} + f\cos\omega t. \qquad (14.18)$$

This is similar to the chaotic pendulum we studied, but has some advantage in the ease with which the multiple attractor sets can be found. It has been studied by [M&L 85].

3. **Lorenz attractor:** In 1962 Lorenz was looking for a simple model for weather prediction and simplified the heat-transport equations to the

three equations [Tab 89]:

$$\frac{dx}{dt} = 10(y - x), \qquad (14.19)$$
$$\frac{dy}{dt} = -xz + 28x - y, \qquad (14.20)$$
$$\frac{dz}{dt} = xy - \tfrac{8}{3}z. \qquad (14.21)$$

The solution of these simple nonlinear equations gave the complicated behavior that has led to the modern interest in chaos (after considerable doubt regarding the numerical solutions).

4. **A 3-D computer fly:** Plot, in 3-D space, the equations

$$x = \sin ay - z\cos bx, \qquad (14.22)$$
$$y = z\sin cx - \cos dy, \qquad (14.23)$$
$$z = e\sin x. \qquad (14.24)$$

Here the parameter e controls the degree of randomness.

5. **Hénon–Heiles potential:** The potential and Hamiltonian

$$V(x,y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + x^2y - \tfrac{1}{3}y^3, \qquad (14.25$$
$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + V(x,y), \qquad (14.26$$

are used to describe three astronomical objects interacting. They bin the objects near the origin, but releases them if they move far out The equations of motion for this problem follow from the Hamiltonia equations

$$\frac{dp_x}{dt} = -x - 2xy, \qquad \frac{dp_y}{dt} = -y - x^2 + y^2, \qquad (14.27$$
$$\text{where} \quad \frac{dx}{dt} = p_x, \qquad \frac{dy}{dt} = p_y. \qquad (14.28$$

(a) Numerically solve for the position $[x(t), y(t)]$ for a particle in th Hénon–Heiles potential.

(b) Plot $[x(t), y(t)]$ for a number of initial conditions. Check that th initial condition $E < \frac{1}{6}$ leads to a bounded orbit.

(c) Produce a Poincare section in the (y, p_y) plane. Plot (y, p_y) ea time an orbit passes through $x = 0$.