Analytic topologies over countable sets

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Abstract

In this article we attempt to a systematic study of analytic topologies over the natural numbers $\mathbf{N}$ (or any countable set X).

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1 Introduction

In this article we attempt to a systematic study of analytic topologies over the natural numbers $\mathbf{N}$ (or any countable set X). We can identify every subset of $\mathbf{N}$ with its characteristic function, so its power set $\mathcal{P}(\mathbf{N})$ is identified with the Cantor space $2^{\mathbf{N}}$. Since every topology over $\mathbf{N}$ is a subset of $\mathcal{P}(\mathbf{N})$, it is clear then what we mean by saying that τ is closed, open, G_δ, Borel, analytic, etc. That this kind of restriction on a topology shows up in purely topological results is perhaps not as widely known as it should. For example, it shows up in Godefroy's characterization of separable compacta K that can be embedded in the first Baire class equipped with the topology of pointwise convergence (see [10] and 6.3 below). Namely, this happens exactly when the uniformity K induces on any of its countable dense subsets is analytic. It is perhaps not surprising that many of the examples of countable topological spaces found in the literature are analytic. For example, Arens space ([1]) or its more general version, the Arhangel'skii-Franklin space ([2]), have analytic topologies (see also §5 below). Questions involving convergence in topology are frequently questions about countable spaces with analytic topologies. This is particularly true about spaces appearing as subspaces of some function space. The realization that they are analytic can sometimes be a powerful tool when dealing with these kind of questions (see, for example, theorem 6.6). One of the goals of this article is to make this connections between descriptive set theoretic properties and purely topological properties of a given space more explicit.

On the other hand, there are many results concerning the descriptive set theoretic properties of families of subsets of $\mathbf{N}$, like ideals and filters (see [7, 12, 16, 20, 24, 26]). Every filter has naturally associated a topology, hence those results about the existence of Borel or analytic filters (or ideals) over $\mathbf{N}$ immediately provide examples of topologies over $\mathbf{N}$ of the same (Borel, projective) complexity. These topologies are not Hausdorff, however, given a filter $\mathcal{F}$ over $\mathbf{N}$ by an elementary construction it is easy to define a Hausdorff topology of the same complexity as the filter $\mathcal{F}$. It is known that every G_δ filter is necessarily closed, but there are filters (and hence Hausdorff topologies) in all levels of the Borel hierarchy above the third level.

The paper is organized as follows. In §2 we analyze closed and G_δ topologies. We will also look at topologies that have the Baire property and show that if a T_1 topology on X is a Baire measurable subset of 2^X then it must be in fact meager unless it has only finitely many limits points. This is an analog of the well known fact that every analytic ideal (containing all finite sets) is meager. In §3 we present some results concerning the complexity of bases and subbases. In §4 we analyze the complexity of Hausdorff topologies. In §5 we present some critical examples of analytic topologies of various complexities. In §6 we show that every analytic regular topology is homeomorphic to a countable subspace of the function space $C_p(\mathbf{N}^{\mathbf{N}})$. This result naturally leads to the notion of a Rosenthal compactification of an countable analytic space.

The last three sections are devoted to the study of the ideal of nowhere dense sets $NWD(\tau)$, where τ is a given topology over X. One of the questions we address is the following: given a Borel (analytic) ideal $\mathcal{I}$ over X, what are the possible topologies τ such that $\mathcal{I} = NWD(\tau)$? We classify under equivalence the family $NWD(\tau)$ when τ is an Alexandroff topology over $\mathbf{N}$. We show that $NWD(\tau)$ is not a p-ideal for τ analytic. Many of the structural properties of ideals over $\mathbf{N}$ have been established by using two important notions for comparing ideals: Tukey reducibility [8] and Rudin-Blass reducibility. We analyze the ideal of nowhere dense sets from these points of view.

Some preliminary results concerning the problems studied in this paper appeared in [25].

We will use the standard notions and terminology of descriptive set theory (see for instance [11]). X will always denote a countable set. $\omega^{<\omega}$ denotes the collection of finite sequences of natural numbers. If $s \in \omega^{<\omega}$ and $n \in \mathbf{N}$ then $s\widehat{\ }(n)$ is the concatenation of s with n. Let A, B be subsets of topological spaces Y and Z respectively: As usual $A \leq_W B$ denotes the fact that A is Wadge reducible to B, that is to say, there is a continuous function $f : Y \to Z$ such that $x \in A$ iff $f(x) \in B$. The ideal of finite subsets of $\mathbf{N}$ is denoted by FIN, $\emptyset \times \text{FIN}$ denotes the ideal over $\mathbf{N} \times \mathbf{N}$ given by $A \in \emptyset \times \text{FIN}$ iff for all n, $\{i : (n, i) \in A\}$ is finite and $\text{FIN} \times \emptyset$ denotes the ideal given by $A \in \text{FIN} \times \emptyset$ iff there is n such that $A \subseteq n \times \mathbf{N}$, where as usual we identify a natural number n with the set $\{0, \cdots, n-1\}$.

2 Closed and G_δ topologies

In this section we will analyze over a countable set X topologies that are closed or G_δ as subset of 2^X.

We first recall some notions. A topology τ over X is said to be *Alexandroff* if it is closed under arbitrary intersection, equivalently, if $N_x = \bigcap\{V : x \in V \text{ and } V\ \tau\text{-open}\}$ is τ-open for every $x \in X$. N_x is called the minimal neighbourhood of x. It is well known that Alexandroff topologies are represented by quasi-orders as given by the following theorem:

Theorem 2.1. *A topology τ over X is Alexandroff iff there is a binary relation $\leq_\tau$ over X which is transitive and reflexive and such that $A \in \tau$ iff for every $x \in A$ we have $\{y \in X : x \leq_\tau y\} \subseteq A$. Moreover, the minimal neighbourhood of x is $\{y \in X : x \leq_\tau y\}$. Furthermore, τ is T_0 iff $\leq_\tau$ is antisymmetric (i.e. $\leq_\tau$ is a partial order). Also, $cl_\tau(A) = \bigcup_{x \in A} cl_\tau(\{x\}) = \bigcup_{x \in A}\{y \in X : y \leq_\tau x\}$. Thus $\leq_\tau$ is given by $y \leq_\tau x$ iff $y \in cl_\tau(\{x\})$.* $\square$

We start by considering the question of when a given topology τ over a countable set X is an open, closed or dense subset of 2^X.

Theorem 2.2. *Let τ be a topology over X.*

(i) $\tau \subseteq 2^X$ is closed if, and only if τ is Alexandroff.

(ii) $\tau \subseteq 2^X$ *is open if, and only if there is a* τ*-clopen, discrete and co-finite subset of* X*. In particular, every open topology is clopen.*

(iii) The closure of τ *in* 2^X*, denoted by* $\overline{\tau}$*, is a topology. Therefore* $\overline{\tau}$ *is the smallest Alexandroff topology containing* τ*.*

(iv) τ *is dense in* 2^X *if, and only if* τ *is* T_1*.*

Proof: First, it is not difficult to show that if $S \subseteq 2^X$ is a closed set which is closed under finite intersections (resp. unions), then S is closed under arbitrary intersections (resp. unions). From this (iii) follows, since $\overline{\tau}$ is a closed set closed under finite intersection and unions. Also from this observation half of (i) easily follows. For the other half of (i), let τ be an Alexandroff topology and A_n a sequence of τ-open sets converging (pointwise) to A. If $x \in A$, then N_x, the minimal neighbourhood of x, is a subset of eventually every A_n and therefore a subset of A. Hence A is open. For (ii), let τ be an open topology, then $\emptyset$ and X are interior points of τ. Then, it is not hard to see that there is a finite set F such that F is τ-clopen and $X - F$ is discrete. From this it follows that τ is clopen. Finally, for (iv) let us suppose that τ is dense in 2^X. Let A_n be a sequence of open sets converging pointwise to $\{x\}$. Let $y \neq x$, then there is n such that $x \in A_n$ and $y \notin A_n$. Hence $\{y\}$ is closed. Conversely, suppose τ is T_1. Then the collection of τ-closed sets contains all finite sets and hence it is dense in 2^X. Since the map $A \mapsto X - A$ is a homeomorphism then τ has to be also dense. $\square$

The simplest example of an F_σ topology is the co-finite topology. Given a filter $\mathcal{F}$ over ω, we will identify $\mathcal{F}$ with the topology $\mathcal{F} \cup \{\emptyset\}$. Since filters and ideals are dual objects, we will also identify an ideal with the topology associated with its dual filter. Nice examples of F_σ ideals can be found in [16]. Next we give an elementary method to construct a Hausdorff topology based on a filter, it will be used to give examples in the sequel.

Example 2.3. Let $\mathcal{F}$ be a filter over ω. We define a topology $\tau(\mathcal{F})$ over $\omega + 1$ by $\tau(\mathcal{F}) = \{\{\omega\} \cup A : A \in \mathcal{F}\} \cup \mathcal{P}(\omega)$. It is clear that if $\mathcal{F}$ is non principal then $\tau(\mathcal{F})$ is a Hausdorff topology. Since the function $f : 2^\omega \to 2^{\omega+1}$ given by $f(A) = A \cup \{\omega\}$ is continuous and $A \in \mathcal{F}$ iff $f(A) \in \tau(\mathcal{F})$, then $\mathcal{F}$ is Wadge reducible to $\tau(\mathcal{F})$. Also notice that if $\mathcal{F}$ is a non trivial filter, then ω is the only limit point of $(\omega + 1, \tau(\mathcal{F}))$. In fact, it is clear that this is a characterization of such spaces. We state this observations in the next proposition for later reference.

Proposition 2.4. *(i) For every filter* $\mathcal{F}$, $\tau(\mathcal{F})$ *is a Hausdorff topology and* $\mathcal{F} \leq_W \tau(\mathcal{F})$.

(ii) Let (X, τ) *be a Hausdorff space such that* $X^{(1)} = \{x_1, \cdots, x_n\}$*. Then there is a partition of* X *in finitely many clopen pieces* $X_1, \cdots, X_n$ *with* $x_i \in X_i$ *and there are non principal filters* $\mathcal{F}_i$ *over* $X_i - \{x_i\}$ *for* $1 \leq i \leq n$ *such that* (X, τ) *is homeomorphic to* $\oplus_1^n (X_i, \tau(\mathcal{F}_i))$*. In fact, the filters are given by* $\mathcal{F}_i = \{A \subseteq (X_i - \{x_i\}) : A \cup \{x_i\} \in \tau\}$*, thus* $\mathcal{F}_i \leq_W \tau$*.* $\square$

Since every G_δ filter is necessarily principal (and hence closed), then 2.4 does not provide examples of G_δ topologies. In fact the situation is quite different. We show below that there are no non-discrete T_1 topologies over $\mathbf{N}$ that are G_δ as subsets of $2^{\mathbf{N}}$, and later we give an example of a G_δ-complete T_0 topology. But first we will address the question of when a given topology τ over X is a meager subset of 2^X. The next result is interesting by itself.

Theorem 2.5. *Let* G *be a comeager subset of* $2^{\mathbf{N}}$*. If* G *is closed under finite unions and intersection then* $G = 2^{\mathbf{N}}$*.*

Proof: First we recall that $2^{\mathbf{N}}$ is a Polish group (i.e. a topological group such that its topology is separable and completely metrizable) with symmetric difference as the group operation (it is the countable product of the group $\{0, 1\}$ with addition modulo 2).

Let G be a comeager subset of $2^{\mathbf{N}}$ which is closed under finite unions and intersections. Let $CL(G) = \{A \in 2^{\mathbf{N}} : A, A^c \in G\}$, then $CL(G)$ is a subgroup of the Cantor group $2^{\mathbf{N}}$. On the other hand, since G is comeager then $CL(G) = G \cap \{\mathbf{N} - A : A \in G\}$ is also comeager (since $A \mapsto \mathbf{N} - A$ is a homeomorphism). Now note that a comeager subgroup of $2^{\mathbf{N}}$ must in fact be equal to $2^{\mathbf{N}}$ (see for instance, I.9.11 of [11]). $\square$

Corollary 2.6. *If a T_1 topology τ is a Baire-measurable subset of 2^X then it must be in fact meager subset of 2^X unless it has only finitely many nonisolated points.*

Proof: Suppose τ is not meager. Let Let K, F finite disjoint subsets of $\mathbf{N}$ such that τ is comeager in the basic nbhd V given by $\{A \subseteq \mathbf{N} : K \subseteq A \;\; \& \;\; A \cap F = \emptyset\}$. Let $B = \mathbf{N} - (K \cup F)$. Let ρ be the restriction of τ to B. Then ρ is comeager in 2^B. Hence by 2.5 ρ is the discrete topology. Hence the limit points of τ belongs to $K \cup F$ and therefore there are only finitely many of them. $\square$

Corollary 2.7. *If a T_1 topology on a countable set X is a G_δ subset of 2^X, then it must be discrete.*

Proof: Just notice that since τ is T_1 then by 2.2(iv) τ must be a dense subset of 2^X. $\square$

Remark 2.8. There are topologies with infinitely many limit points which are not meager. For instance, consider $\tau = \{A \subseteq \mathbf{N} : 0 \in A\} \cup \{\emptyset\}$. Then τ is an Alexandroff T_0 topology, 0 is the only isolated point and τ contains a basic open set.

There are some simple $\mathbf{\Delta}^0_2$ topologies over $\mathbf{N}$ (i.e., they are both G_δ and F_σ). For instance, let $X = \omega+1$ with the usual order and τ be the corresponding Alexandroff topology. Let $\tau' = \tau - \{\{\omega\}\}$. Then it is easy to check that $\overline{\tau'} = \tau$ and also that τ' is Δ^0_2, i.e., it is both F_σ and G_δ. The next example shows that there are true G_δ topologies.

Example 2.9. *A T_0 topology on a countable set X which is a G_δ-complete subset of 2^X.*

We first show a general result that points to a natural place where to look for G_δ topologies.

Claim 1: Let τ be an Alexandroff topology over a countable set X and let $D(\tau) = \{A \in \tau : A \text{ is } \tau\text{-dense}\}$ and $\rho = D(\tau) \cup \{\emptyset\}$. Then ρ is a G_δ topology. Moreover, if τ has no isolated points then $\tau = \overline{\rho}$.

Proof: It is straightforward to check that $A \in D(\tau)$ iff for all $x \in X$ there is $y \in A$ such that $x \leq_\tau y$, where $\leq_\tau$ is the order given by 2.1. So $D(\tau)$ is G_δ and so is ρ. For the second claim observe that τ has no isolated points if, and only if every finite set is τ-nowhere dense. We will show that $\tau = \overline{\rho}$. Let $O \in \tau$ and F, K disjoint finite sets such that $F \subseteq O$ and $K \cap O = \emptyset$. Let $V = X - \overline{K}$, then by hypothesis V is τ-open dense, $F \subseteq V$ and $V \cap K = \emptyset$. $\square$

In general, the topology given by the previous result is not a true G_δ set. For instance, let $<$ be the usual order on $\omega + 1$ and consider the Alexandroff topology. An open set V is τ-dense iff $\omega \in V$. Hence $D(\tau)$ is closed.

Let $X = 2^{<\omega}$ (the collection of all binary sequences) and let $\preceq$ be the usual extension order. Let τ be the Alexandroff topology over X given by $\preceq$. For each $s \in 2^{<\omega}$ the minimal neighbourhood of s is $N_s = \{t \in 2^{<\omega} : s \preceq t\}$. Let $\rho = D(\tau) \cup \{\emptyset\}$, since τ is a T_0 topology without isolated points then $\tau = \overline{\rho}$ and therefore ρ is also T_0. We will show that ρ is a G_δ-complete subset of $2^{2^{<\omega}}$. To that end, we will show some simple facts that will simplify the arguments.

Claim 2: Let $T \subseteq 2^{<\omega}$, then T is τ-closed if, and only if T is a tree.
Proof: Since τ is an Alexandroff topology, then by 2.1 $cl_\tau(\{s\}) = \{t \in 2^{<\omega} : t \preceq s\}$ and T is τ-closed if, and only if $cl_\tau(\{s\}) \subseteq T$ for all $s \in T$. $\square$

Claim 3: Let T be a binary tree, as usual $[T]$ denotes the set of (infinite) branches of T. Then T is τ-closed-nowhere-dense if, and only if $[T]$ is nowhere dense in $2^{\mathbf{N}}$.
Proof: It is easy to check that for every τ-closed set T and every $s \in 2^{<\omega}$, $U_s = \{\alpha \in 2^{\mathbf{N}} : s \prec \alpha\} \subseteq [T]$ iff $N_s = \{t \in 2^{<\omega} : s \preceq t\} \subseteq T$. $\square$

The following is a well known fact (see [11], page 27): Let $\varphi : \mathcal{K}(2^{\mathbf{N}}) \mapsto 2^{2^{<\omega}}$ given by $\varphi(K) = \{s \in 2^{<\omega} : \exists \alpha \in K \; s \prec \alpha\}$. Then φ is 1-1 and continuous and $\varphi(K)$ is a tree such that $K = [\varphi(K)]$. In fact, φ is a homeomorphism of $\mathcal{K}(2^{\mathbf{N}})$ onto the set of binary pruned trees. Since the collection of nowhere dense closed subsets of $2^{\mathbf{N}}$ is G_δ-complete (see [13]), then from the claims above we conclude that $\{F \subseteq 2^{<\omega} : F \text{ is } \tau\text{-closed-nowhere-dense set}\}$ is also G_δ-complete. Finally, since the complementation function on 2^X is an homeomorphism then it is clear that $D(\tau)$ is G_δ-complete. $\square$

3 Complexity of bases and subbases

We now consider the problem of the complexity of a given topology generated by a closed, F_σ, or analytic base.

The following fact is easy to verify and will be used in the sequel.

Proposition 3.1. *Let $f, g : 2^X \times 2^X \to 2^X$ $h : 2^X \to 2^X$ be the functions defined by $f(A, B) = A \cap B$, $g(A, B) = A \cup B$ and $h(A) = X - A$. Then f, g and h are continuous and open. Moreover, h is a homeomorphism.* $\square$

In particular, the previous results says that for a given topology τ the collection of τ-open sets and τ-closed sets have the same descriptive set theoretic complexity.

Proposition 3.2. *Let (X, τ) be a countable topological space.*

(i) X admits an F_σ base iff it admits an F_σ subbase.

(ii) If X admits an F_σ base (or subbase) then τ is $\mathbf{\Pi}^0_3$. In particular, if τ is second countable topology, then τ is $\mathbf{\Pi}^0_3$.

(iii) If X admits a $\mathbf{\Sigma}^1_1$ base (or subbase) then τ is $\mathbf{\Sigma}^1_1$.

(iv) Suppose X is Hausdorff and has an F_σ base. If $X^{(1)}$ (the set of limit points) is finite, then τ is F_σ.

(v) If X is T_1 and non discrete, then τ does not have a closed base.

Proof: Let $\mathcal{B}$ be a base for τ, then we have

$$A \in \tau \iff \forall x\, [\, x \in A \to \exists\, B \in \mathcal{B}\, (\, x \in B \,\&\, B \subseteq A\,)\,] \tag{1}$$

If $\mathcal{B}$ is F_σ (resp. $\mathbf{\Sigma}^1_1$), then from (1) it follows that τ is $\mathbf{\Pi}^0_3$ (resp. $\mathbf{\Sigma}^1_1$). If $\mathcal{S}$ is an F_σ subbase for τ then it is easy to check using 3.1 that the base generated by $\mathcal{S}$ is also F_σ. This shows (i), (ii) and (iii). (iv) follows from 2.4(ii), since the filters $\mathcal{F}_i$ given there are clearly generated by an F_σ set and therefore they must be F_σ. Hence τ is F_σ. To see (v) suppose that F is a closed base for τ and fix $x \in X$. For each finite set $A \subseteq X$ with $x \notin A$, there is $V_A \in F$ such that $x \in V_A \subseteq X - A$. Since $\{V_A\}$ converges to $\{x\}$ and F is closed, then τ is the discrete topology. □

Remark 3.3. (1) There are Hausdorff topologies such that $X^{(1)}$ is finite but τ is not F_σ (and of course τ does not have an F_σ base). For instance, let $\mathcal{F}$ be a filter over ω which is not F_σ (for example, the dual filter of $\emptyset \times \mathrm{FIN}$). Then $\tau(\mathcal{F})$ (defined in 2.3) is $\mathbf{\Pi}^0_3$-complete, but $X^{(1)} = \{\omega\}$.

(2) There are $\mathbf{\Pi}^0_3$ topology without an F_σ base (or even subbase). In fact, let τ be the topology associated with the ideal $\emptyset \times \mathrm{FIN}$. First, notice that if $\mathcal{B}$ is F_σ then $\mathcal{B}^{mon} = \{A : \exists B \in \mathcal{B}\ A \subseteq B\}$ is also F_σ. Now, if $\mathcal{B}$ is a base for τ (w.l.o.g we assume $\emptyset \notin \mathcal{B}$), then it is easy to check that $\tau = \mathcal{B}^{mon} \cup \{\emptyset\}$ (the fact that τ is not Hausdorff is irrelevant, since by a similar argument if $\mathcal{F}$ is the dual filter of $\emptyset \times \mathrm{FIN}$ (identifying $\omega \times \omega$ with ω), then $\tau(\mathcal{F})$ does not admit an F_σ base). A more interesting example will be given later. A natural question is to determine which $\mathbf{\Pi}^0_3$ topologies admit an F_σ base. □

Theorem 3.4. *Every Hausdorff topology on a countable set generated by a F_σ subbase has in fact a closed subbase.*

Proof: Let $\{x_i\}_{i=1}^\infty$ be an enumeration of X. For each n we fix an open neighbourhood V_n of x_n such that $x_i \notin \overline{V_n}$ for all $i < n$. Let $K = \bigcup_{n=1}^\infty K_n$ be a fixed base for X such that each K_n is closed and $K_n \subseteq K_{n+1}$ for all n. For $n \geq 1$ set

$$\widehat{K}_n = \{A \cup (X \setminus \overline{V_i}) \cup \bigcup_{l=i+1}^{n} V_l :\ \ 1 \leq i \leq n,\ A \in K_n,\ x_i \in A\}$$

Clearly each $\widehat{K}_n$ is a closed set of open subsets of X. Let $\widehat{K} = \bigcup_n \widehat{K}_n$. We claim that $\widehat{K} \cup \{X\}$ is closed in 2^X. It suffices to show that every sequence $B_k \in \widehat{K}_{n_k}$ ($k \in \mathbf{N}$) such that $\{n_k\}$ is strictly increasing accumulates to X. So let F be a finite subset of X and let k_0 be such that $F \subseteq \{x_i : 1 \leq i \leq n_{k_0}\}$. Consider B_k for $k \geq k_0$. Then B_k is of the form

$$A_k \cup (X \setminus \overline{V_{i_k}}) \cup \bigcup_{l=i_k+1}^{n_k} V_l$$

for some $i_k \in \{1, \cdots, n_k\}$. Consider $x \in F$. If $x = x_{i_k}$, then $x \in A_k \subseteq B_k$. If $x = x_i$ for $i < i_k$, then $x \in (X \setminus \overline{V_{i_k}}) \subseteq B_k$. If $x = x_i$ for $i \in \{i_{k+1}, \cdots, n_k\}$, then $x \in V_i \subseteq B_k$. This shows that $F \subseteq B_k$. Let

$$\mathcal{B}_n = \{A \cap V_i \cap \bigcap_{l=i+1}^{n} (X \setminus \overline{V_l}) :\ 1 \leq i \leq n,\ A \in K_n,\ x_i \in A\}$$

It is clear that $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ forms a basis of X. Note that a given element $A \cap V_i \cap \bigcap_{l=i+1}^{n}(X \setminus \overline{V_l})$ of $\mathcal{B}_n$ is equal to the intersection of the element $A \cup (X \setminus \overline{V_i}) \cup \bigcup_{l=i+1}^{n} V_l$ of $\widehat{K}_n$ with V_i and each $X \setminus \overline{V_l}$ $(l = i+1, \cdots, n)$. Note that the sequence $\{V_n\}$ converges to $\emptyset$ while the sequence $\{X \setminus \overline{V_n}\}$ converges to X. It follows that

$$\{X, \emptyset\} \;\cup\; \bigcup_n \widehat{K}_n \;\cup\; \{V_n\}_{n=1}^{\infty} \;\cup\; \{X \setminus \overline{V_n}\}_{n=1}^{\infty}$$

is a closed subset of 2^X and it forms a subbasis of X. □

Remark 3.5. Note that the above proof shows that every Hausdorff second countable space has a subbasis which is closed and countable. Thus, in particular, the topology of the rationals is generated by a countable closed subbase. In fact the above proof shows that the topology of the rationals is generated by a closed set with only two non-isolated points (i.e. the union of two converging sequences). A natural question that remains unanswered asks whether a regular topology with an F_σ base has an F_σ base consisting of clopen sets. Note that the proof of Theorem 3.4 can turn any F_σ base consisting of clopen sets into a closed subbase consisting of clopen sets. □

Improving an earlier result of Zafrany [26], Solecki and the first author have recently (see [22]) shown that every analytic filter is generated by a G_δ subset. This suggests that similar facts might be true for analytic topologies on a countable set. The following result of Solecki [21], included here with his permission, goes along these lines.

Theorem 3.6. *Let τ be an analytic topology on a countable set X. Suppose there is a sequence $\{U_n\}$ of open sets such that $\bigcap_n \overline{U_n} = \emptyset$ and $\tau|U_n$ is uncountable for all n. Then τ has a $\mathbf{\Sigma}_3^0$ subbase. If additionally τ is T_1, then τ has a G_δ subbase.*

Proof: By the perfect set property of analytic sets, for each n, we can fix $Z_n \subseteq \tau|U_n$ that is homeomorphic to $\mathbf{N}^{\mathbf{N}}$. Then for each n we fix a continuous surjection $f_n : Z_n \to \tau|(X \setminus \overline{U_n})$. Define

$$Z = \{X - \overline{U_n} :\ n \in \omega\} \cup \{V \cup f_n(V) :\ n \in \omega, V \in Z_n\}$$

Note that, for each n, the set $Z_n^* = \{V \cup f_n(V) :\ V \in Z_n\}$ is homeomorphic to Z_n, so it is G_δ in 2^X. Hence Z is $\mathbf{\Sigma}_3^0$. To see that Z is a subbase of τ, note that for all $x \in X$ there is n such that $x \in X \setminus \overline{U_n}$. Let $U \subseteq X \setminus \overline{U_n}$ be an open set with $x \in U$. Find $V \in Z_n$ with $f_n(V) = U$. Then $V \cup U \in Z$. Since $V \subseteq U_n$, then $U = (X \setminus \overline{U_n}) \cap (V \cup U)$.

If τ is T_1, enumerating X as $\{x_n\}$ and reenumerating $\{U_n\}$ we may assume that $x_n \notin \overline{U_n}$ and $x_i \notin U_n$ for $i < n$. Also we will assume that f_n has range equal to the collection of open subsets of $X \setminus (\overline{U_n} \cup \{x_i : i < n\})$. The definition of Z remains the same except that we put the sets $X \setminus (\overline{U_n} \cup \{x_i : i < n\})$ in place of $X \setminus \overline{U_n}$. Note that this sequence of sets converges to $\emptyset$. So it remains only to show that the union Z^* of the corresponding collection of sets Z_n^* is G_δ. To see this, note that $W \notin Z^*$ iff $W \notin Z_n^*$ for $n = min\{i : x_i \in W\}$. □

Corollary 3.7. *Every analytic T_2 topology has a G_δ subbase.*

Proof: If every point $x \in X$ has a neighbourhood V_x such that $U_x = X \setminus \overline{V_x}$ is infinite, the sequence $\{U_x\}$ satisfies the hypothesis of 3.6. Otherwise, X would be either finite or it would contain only one nonisolated point x_∞ such that every neighbourhood of x_∞ is cofinite in X. In the later case, X would homeomorphic to $\omega + 1$ with the order topology and hence by 3.4 it has a closed subbase. □

Remark 3.8. Note that these results still leave it unclear whether every analytic topology on a countable set has a Borel base or subbase. Of course, if the answer is positive one would then like to determine the minimal Borel complexity of such base or subbase. $\square$

4 Complexity of Hausdorff topologies

In this section we consider the complexity of analytic T_2 topologies having infinitely many limit points. The following general fact shows that they all are at least $\mathbf{\Pi}^0_3$. Notice that the topology of a convergent sequence in a metric space is an example of an F_σ Hausdorff topology with finitely many limit points (see also 3.2).

Theorem 4.1. *Let τ be an analytic T_2 topology over a countable X such that $X^{(1)}$ is infinite. Then $\emptyset \times FIN \leq_W \tau$.*

Corollary 4.2. *Every T_2 topology over a countable set with an F_σ base and infinitely many non-isolated points is $\mathbf{\Pi}^0_3$-complete.* $\square$

Corollary 4.3. *The topology of the rationals is $\mathbf{\Pi}^0_3$-complete.* $\square$

The proof of 4.1 will need the following general fact.

Proposition 4.4. *Let τ be a T_1 analytic topology with an infinite cellular family (a family of pairwise disjoint sets) of non-discrete open sets. Then $\emptyset \times FIN \leq_W \tau$; in particular, τ is $\mathbf{\Pi}^0_3$-hard.*

Proof: Let $\{V_i\}$ be a fixed cellular family of non-discrete of τ-open sets. For each i fix a non isolated point $x_i \in V_i$. Let $\mathcal{F}_i$ be the restriction of the neighbourhood filter of x_i to $V_i - \{x_i\}$. Then $\mathcal{F}_i$ is a proper analytic filter on an infinite set, so by a well known result of Mathias (see [15]), we can find a sequence $\{F^i_n\}_{n=0}^\infty$ of pairwise disjoint finite subsets of $V_i - \{x_i\}$ such that for every infinite $M \subseteq \mathbf{N}$, $\bigcup_{n\in M} F^i_n$ accumulates to x_i. Define $f : 2^{\mathbf{N}\times\mathbf{N}} \to 2^X$ by

$$f(A) = \bigcup_{i=0}^{\infty} \bigcap_{(i,n)\in A} (V_i - F^i_n)$$

Clearly f is continuous. It is also not hard to check that a subset $A \subseteq \mathbf{N}\times\mathbf{N}$ belongs to $\emptyset \times \mathrm{FIN}$ if, and only if, $f(A)$ is τ-open. $\square$

The following lemma gives some sufficient conditions for having the hypothesis of 4.4.

Lemma 4.5. *Let (X, τ) be a Hausdorff space such that $X^{(1)}$ is infinite. Then any of the following conditions implies that there is an infinite cellular family of non-discrete τ-open sets.*

(i) $X^{(2)} \neq \emptyset$.

(ii) (X, τ) *is regular.*

Proof: (i) Suppose $X^{(2)} \neq \emptyset$. Let $x \in X^{(2)}$ and $y_1 \neq x$ with $y_1 \in X^{(1)}$. Let W and V_1 be disjoint open sets containing x and y_1 respectively. Then $W \cap X^{(1)} \neq \emptyset$. Let $y_2 \in W$ be a limit point. We can repeat the construction inside W and find V_2 with $y_2 \in V_2$. In this way we construct a sequence of limit points $\{y_n\}$ and pairwise disjoint open sets $\{V_n\}$ with $y_n \in V_n$.

(ii) If τ is zero-dimensional (i.e., it admits a base of clopen sets), $X^{(1)}$ is infinite and $X^{(2)} = \emptyset$, then such family of open sets exists. In fact, we can define by induction a collection $\{W_x : x \in X^{(1)}\}$ of pairwise τ-clopen sets with $x \in W_x$. If τ is regular, $X^{(1)}$ is infinite and $X^{(2)} = \emptyset$, then τ is zero-dimensional. In fact, let $x \in X^{(1)}$ and V be an open set such that $x \in V$ and $X^{(1)} \cap V = \{x\}$. Then by regularity, there is $W \subseteq V$ open such that $x \in W$ and $cl_\tau(W) \subseteq V$. Then $cl_\tau(W) \cap X^{(1)} = \{x\}$, thus W is clopen. $\square$

The following example shows that some assumptions in 4.5 are needed.

Example 4.6. *There is a second countable Hausdorff topology τ on a countable set X such that $X^{(1)}$ is infinite but (X, τ) has no infinite cellular families of non discrete open sets.*
To see this, fix an independent family A_s ($s \in \mathbf{N} \times \mathbf{N}$) of infinite subsets of $\mathbf{N}$, i.e., a family with the property that

$$(\bigcap_{s \in E} A_s) \cap (\bigcap_{t \in F} (\mathbf{N} - A_t)$$

is infinite for every pair E and F of disjoint finite subsets of $\mathbf{N} \times \mathbf{N}$. Let $X = \mathbf{N} \times 2$ with points of $\mathbf{N} \times \{0\}$ all isolated while neighbourhoods of some $(n, 1)$ are of the form

$$U^F_{(n,1)} = \{(n, 1)\} \ \cup \ [(\bigcap_{i<n} A_{(i,n)}) \cap (\bigcap_{j \in F} (\mathbf{N} - A_{(n,j)}))] \times \{0\},$$

where F is a finite subset of $\mathbf{N} - \{0, \cdots, n\}$. Note that for $n < m$, $U^{\{m\}}_{(n,1)}$ and $U^{\emptyset}_{(m,1)}$ are two disjoint neighbourhoods of $(n, 1)$ and $(m, 1)$ respectively, so τ is T_2. Note also that by the independence of the family $A_{(n,m)}$ ($(n, m) \in \mathbf{N} \times \mathbf{N}$), the closure of every $U^F_{(n,1)}$ contains $(m, 1)$ for all $m > \max(F)$, so there are no infinite cellular families of non-discrete open sets. $\square$

Proof of 4.1: From 4.5 we can assume that $X^{(2)} = \emptyset$, $X^{(1)}$ is infinite and there are no cellular families of non-discrete open subsets of X. Given a closed subspace Y of X it is easy to check that $\tau|Y \leq_W \tau$ and since we are working towards proving that $\emptyset \times \mathrm{FIN} \leq_W \tau$, we can assume also that every such Y has no an infinite cellular family of non-discrete (relatively) open sets, as far as $Y^{(1)}$ is infinite. In this context we make the following

Claim: Let Y be a closed subspace of X such that $Y^{(1)}$ is infinite. Then for every $y \in Y^{(1)}$ there is an open neighbourhood U of y such that the closed subspace $Z = Y - U$ has the property that $Z^{(1)}$ is infinite.
Proof of the claim: Otherwise, for every finite sequence $y_1, \cdots, y_{k-1}$ of elements of $Y^{(1)} - \{y\}$ and every sequence of open sets $U_0, \cdots, U_{k-1}$ such that $y_i \in U_i$ and $y \notin \overline{U_i}$, for all $i < k$, the set $C_{k-1} = \bigcup_{i<k} \overline{U_i}$ being a complement of a neighbourhood of y, can contain only finitely many points from $Y^{(1)}$. So, we can choose another point $y_k \in Y^{(1)} - \{y\}$ not in C_{k-1} and a neighbourhood U_k of y_k disjoint from C_{k-1} such that $y \notin \overline{U_k}$. Proceeding this way, we can construct a cellular family of non-discrete open subsets of Y, contradicting our assumption. $\square$

Let $\{z_n\}$ enumerate $X^{(1)}$. We will define by induction an increasing sequence n_k of integers, a sequence $\{O_k\}$ of open sets and a sequence of finite sets $\{F^k_n\}$ such that

(1) $z_{n_k} \in O_k$ for all k.

(2) $\{F_n^k\}$ is a sequence of pairwise disjoint finite sets of isolated points in O_k and $z_{n_k} \in \overline{\bigcup_{n\in A} F_n^k}$ for all infinite $A \subseteq \mathbf{N}$.

(3) $O_k \cap F_n^l = \emptyset$ for all $l > k$ and all n.

(4) $Z_k = X - (\bigcup_{i=0}^k O_i)$ is a closed subspace such that $Z_k^{(1)}$ is infinite and n_{k+1} is the minimal integer n such that $z_n \in Z_k^{(1)}$.

By the claim there is an open neighbourhood O_0 of z_0 such that $Z_0 = X - O_0$ is a closed subspace with the property that $Z_0^{(1)}$ is infinite. Let $\mathcal{F}$ be the neighbourhood filter of z_0 restricted to $X^{(0)} \cap O_0$. By the theorem of Mathias, already used above, there is a sequence of $\{F_n^0\}$ of pairwise disjoint finite subsets of $X^{(0)} \cap O_0$ such that (2) holds. Let $n_0 = 0$ and n_1 be the minimal n such that $z_n \in Z_0^{(1)}$. It is clear that $(1), \cdots, (4)$ hold for $k = 0$.

For the inductive step, suppose we have defined n_i for $i \leq k+1$ and $\{O_i\}$ and $\{F_n^i\}_{n=0}^\infty$ for $i \leq k$ such that $(1), \cdots, (4)$ hold. By the claim there is an open neighbourhood O_{k+1} of $z_{n_{k+1}}$ such that $Z_{k+1} = Z_k - O_{k+1}$ is a closed subspace such that $Z_{k+1}^{(1)}$ is infinite. Let n_{k+2} be the least integer n such that $z_n \in Z_{k+1}^{(1)}$, so (1) and (4) holds. By the theorem of Mathias applied to the neighbourhood filter of $z_{n_{k+1}}$ restricted to $X^{(0)} \cap Z_k \cap O_{k+1}$ there is a sequence $\{F_n^{k+1}\}$ of pairwise disjoint finite subsets of $X^{(0)} \cap Z_k \cap O_{k+1}$ such that (2) and (3) hold.

Define $f : 2^{\mathbf{N}\times\mathbf{N}} \to 2^X$ as before:

$$f(A) = \bigcup_{(k,n)\in A} F_n^k$$

Since the sets F_n^k $(k, n \in \mathbf{N})$ are finite and pairwise disjoint (from (2) and (3)) then f is continuous. To see that f is a reduction of $\emptyset \times \mathrm{FIN}$ to the collection of τ-closed sets, suppose that $A \notin \emptyset \times \mathrm{FIN}$, then there is k such that the vertical section A_k is infinite, so by (2) $z_{n_k} \in f(A)$ and thus $f(A)$ is not closed. On the other hand, suppose $A \in \emptyset \times \mathrm{FIN}$ and $z_n \notin f(A)$. Let k be the least integer such that $n_k \leq n < n_{k+1}$. It is easy to verify using (1) and (4) that $W = O_0 \cup \cdots O_k \cup \{z_n\}$ is an open neighbourhood of z_n. Since each F_n^k is finite from (3) we have that $W \cap f(A)$ is finite. Thus $f(A)$ is closed (actually, it is clopen). $\square$

5 Some examples

We will present examples of topologies of various complexities.

Example 5.1. Let $\mathcal{F}$ be a filter over $\mathbf{N}$ containing the filter of cofinite sets. Define a topology over $X = \omega^{<\omega}$ as follows:

$$U \in \tau_{\mathcal{F}} \Leftrightarrow \{n \in \mathbf{N} : s\widehat{\ }n \in U\} \in \mathcal{F} \text{ for all } s \in U$$

It is clear that $\tau_{\mathcal{F}}$ is T_2, zero dimensional and has no isolated points. From the definition of $\tau_{\mathcal{F}}$ is easy to check that $\tau_{\mathcal{F}}$ is $\mathbf{\Pi}^0_{\alpha+1}$ if $\mathcal{F}$ is $\mathbf{\Pi}^0_{\alpha+1}$ or $\mathbf{\Sigma}^0_\alpha$. On the other hand, consider the function $\phi : 2^{\mathbf{N}} \to 2^X$ given by $\phi(A) = \{\emptyset\} \cup \{s \in \omega^{<\omega} : s(0) \in A\}$. It is clear that ϕ is continuous and $A \in \mathcal{F}$ if and only if $\phi(A) \in \tau_{\mathcal{F}}$. This shows that $\mathcal{F} \leq_W \tau_{\mathcal{F}}$. In particular, if $\mathcal{F}$ is a true $\mathbf{\Pi}^0_\alpha$ set, then so is $\tau_{\mathcal{F}}$. These topologies contains a family of pairwise disjoint open sets U_n such that each U_n is homeomorphic to the entire space X. This explains why the Borel complexity of $\tau_{\mathcal{F}}$ is of the type Π^0_α. It is not difficult to check that there are no T_2 topologies without isolated points such that for a fixed α the relative topology of every nonempty open set is a true $\mathbf{\Sigma}^0_\alpha$ set.

Of special interest is the case of $\tau_{\mathcal{F}}$ when $\mathcal{F}$ is the filter of cofinite sets which we are going to denote simply by τ_{FIN}. We will show that τ_{FIN} does not admit a F_σ base (the same argument applies to $\tau_{\mathcal{F}}$ for any free filter $\mathcal{F}$).

Proposition 5.2. *τ_{FIN} does not admit a F_σ base.*

Proof: We will show some simple claims that will simplify the argument.

Claim 1: Let $A_n \subseteq \omega^n$ be finite and $A = \bigcup_n A_n$. Then A is τ_{FIN}-closed and discrete.
Proof: Let $f(n) = \max\{t(n-1) : t \in A_n\}$ for $n \geq 1$. Let $s \in \omega^k$ and define $U_f = \{s\} \cup \{t \in \omega^{<\omega} : s \prec t \;\&\; t(m-1) > f(m)$ for all $m > k\}$. Notice that U_f is an open set and $U_f \cap A \subseteq \{s\}$. □

Claim 2: Let $K \subseteq \tau_{FIN}$ be a closed set and $s \in \omega^{<\omega}$. Then there is n such that for all $V \in K$ if $s \in V$, then there is $m < n$ such that $s\widehat{\ }m \in V$. Moreover, for all $m > lh(s)$ there is a finite set $A_m \subseteq \omega^m$ such that $s \prec t$ for all $t \in A_m$ and if $V \in K$ and $s \in V$, then $V \cap A_m \neq \emptyset$.
Proof: Otherwise for all n there is $V_n \in K$ such that $s \in V_n$ and $s\widehat{\ }m \notin V_n$ for all $m < n$. We can assume that $V_n \to V \in K$. Then $s \in V$ and $s\widehat{\ }m \notin V$ for all m, which contradicts that V is τ_{FIN}-open. The second claim follows by a simple induction. □

Claim 3: Let $K_n \subseteq \tau_{FIN}$ be closed sets and $s \in \omega^{<\omega}$. Then there is a τ_{FIN}-open neighbourhood O of s such that for all n and all $V \in K_n$ if $s \in V_n$ then $V \not\subseteq O$.
Proof: Fix $s \in \omega^{<\omega}$. For every n such K_n contains an open set V with $s \in V$ pick a finite set $A_n \subseteq \omega^{lh(s)+n}$ as given by claim 2. Let $A = \bigcup_n A_n$ then by claim 1 A is closed and discrete. Let O be the complement of A. Notice that for all n and all $V \in K_n$, if $s \in V$, then $V \cap A_n \neq \emptyset$, thus $V \not\subseteq O$. □

It follows from claim 3 that τ_{FIN} does not have a F_σ base. □

We have already mentioned that $(\omega^{<\omega}, \tau_{\text{FIN}})$ is an homogeneous space. A very interesting description of a space homeomorphic to $(\omega^{<\omega}, \tau_{\text{FIN}})$ where the homogeneity becomes quite transparent is given by van Douwen [6]: Let $A = \{2^n - 1 : n = 0, 1, 2, \cdots\}$. Then A is an infinite subset of $\mathbf{Z}$ which has the property that $0 \in A$ and that $A \cap (k + A)$ is finite for every $z \in \mathbf{Z} \setminus \{0\}$. Let

$$\tau = \{U \subseteq \mathbf{Z} : \ (k + A) \setminus U \text{ is finite for every } k \in U \ \}$$

Then τ is a translation invariant topology on $\mathbf{Z}$ homeomorphic to $(\omega^{<\omega}, \tau_{\text{FIN}})$. Another occurrence of a countable space homeomorphic to $(\omega^{<\omega}, \tau_{\text{FIN}})$ is the space S_ω of Arkhangel'skii and Franklin [2]. So we know that $(\omega^{<\omega}, \tau_{\text{FIN}})$ contains subsets A whose closure require large number of steps of taking sequential closure or in other words, τ_{FIN} is a sequential topology of sequential order equal to ω_1. Yet another occurrence of τ_{FIN} is in the following characterization of the so called Schur property of normed spaces essentially established (though not explicitly stated) in Fremlin [9] (see also [2]).

Theorem 5.3. *The following are equivalent for a normed space E.*
(i) $(\omega^{<\omega}, \tau_{FIN})$ is not embeddable into $(E, weak)$
(ii) Every weakly convergent sequence in E is norm-convergent.

Proof: To see that (ii) implies (i) note that if the subspace $\omega^{\leq 2}$ of τ_{FIN} embeds into $(E, weak)$ via an embedding ψ, then we would have that $\psi(\emptyset)$ is a weak limit of $\psi(\{n\})$ and also that $\psi(\{n\})$'s would be a weak limit of $\{\psi(\{n,m\})\}_{m=n+1}^{\infty}$. By (ii) all these weakly convergent sequences are norm-convergent, thus we can select a diagonal sequence $\{\psi(\{n,m_n\})\}_{n=1}^{\infty}$ which weakly converges to $\psi(\emptyset)$ contradicting the fact that $\{(\{n,m_n\})\}_{n=1}^{\infty}$ is not τ_{FIN}-convergent.

Suppose now that E contains a sequence $\{x_n\}_{n=1}^{\infty}$ of norm 1 vectors which weakly converges to 0. Let $\{t_i\}$ be some natural enumeration of $\omega^{<\omega}$. For $s \in [\omega]^{<\omega}$, put

$$\phi(s) = \sum\{4^i x_j :\ i,j \in \mathbf{N}, t_i \prec t_j \preceq s\}$$

where $\prec$ is the relation of "being an initial segment of ", if $s = \emptyset$ we take $\phi(s) = 0$. Going to a subsequence of $\{x_n\}$ we could have assumed that the x_n's are linearly independent and moreover that some vector $e \in E$ of norm 1 is not in their linear span. Thus we can find a sequence λ_s $(s \in \omega^{<\omega})$ of scalars from $[0,1]$ $(\lambda_\emptyset = 0)$ such that $\psi(s) = \phi(s) + \lambda_s \cdot e$ is one-to-one. This is the mapping that appears in [9] (p. 381) where it is used for showing that (if (ii) fails) the space $(E, weak)$ has sequential order ω_1. However, it is not hard to see that ψ is actually a homeomorphic embedding of $(\omega^{<\omega}, \tau_{\text{FIN}})$ into $(E, weak)$ (compare this with the embedding of S_ω into $(l_2, weak)$ as described in [2] (p. 318).) □

Remark 5.4. A typical normed space with the Schur property (ii) is the space l_1 of absolutely converging series and this is what is frequently called Schur's theorem (see [14] §22). A typical example of a normed space without Schur property is the Hilbert space l_2. This apparently has been first established by von Neumann who proved it by essentially embedding the subspace $\omega^{\leq 2}$ ("the Arens space") of $(\omega^{<\omega}, \tau_{\text{FIN}})$ into l_2. □

Finally we mention a property of τ_{FIN} that makes it clear how far this topology is from being metrizable.

Proposition 5.5. *(see [4], example 3.7) Every continuous map from $(\omega^{<\omega}, \tau_{FIN})$ into a metric space maps a nonempty open set of τ_{FIN} into a point or a nowhere dense set of the metric space.* □

Example 5.6. *A $\mathbf{\Sigma}^1_1$-complete countable group topology.*
We will define for every dense $A \subseteq 2^{\mathbf{N}}$ a topology τ_A on the Boolean group G of all clopen subsets of $2^{\mathbf{N}}$ with symmetric difference as a group operation. The subbase of τ_A are the sets of the form

$$x^+ = \{a \in G : x \in a\},\ x^- = \{a \in G : x \notin a\}$$

where $x \in A$. It is easy to check that if A is analytic the subbase is analytic and therefore so is τ_A. Consider the mapping $f : 2^{\mathbf{N}} \to 2^G$ defined by

$$f(x) = \{a \in G :\ x \in a\}$$

It is not difficult to verify that f is continuous and one-to-one. Finally, observe that if $x \in A$, then $f(x) \in \tau_A$ by definition. On the other hand, if $x \notin A$, then it is not hard to check that $f(x)$ has empty τ_A-interior. This shows that $A \leq_W \tau_A$. For $A = 2^{\mathbf{N}}$, let's denote τ_A by τ_1. The subbase for τ_1 is a compact subset of 2^G so τ_1 is $\mathbf{\Pi}^0_3$-topology in this case. On the other hand, for a carefully chosen analytic non-Borel subset A of $2^{\mathbf{N}}$, then τ_A is a complete $\mathbf{\Sigma}^1_1$-set. Thus a slight change in

$A \subseteq 2^{\mathbf{N}}$ changes the subbasis which can result in a considerable change of the complexity of τ_A. Note that we have actually shown that if A is a true analytic set, then the collection of sets with non empty τ_A-interior is also a true analytic set. This might be a general phenomenon: If τ is a true analytic topology over a countable set X, then $\{C : int_\tau(C) \neq \emptyset\}$ is also analytic and non-Borel. Equivalently, if τ is a true analytic topology, then the collection of τ-dense sets is a true co-analytic set.

It should be clear that all these facts remain true if we restrict ourselves to the subspace H of G consisting of the empty set together with only basic clopen sets $[s] = \{x \in 2^{\mathbf{N}} : \ s \subseteq x\}$, where $s \in 2^{<\omega}$. The point is that now H is topologically a considerably nicer space. For example, (H, τ_1) is a Fréchet space. In fact, first notice that $\emptyset$ is the only non isolated point of (H, τ_1). Hence τ_1 is of the form $\tau(\mathcal{F})$ (as defined in 2.3) for some filter $\mathcal{F}$ over $2^{<\omega} \setminus \{\emptyset\}$. The dual ideal of $\mathcal{F}$ consists of all subsets of $2^{<\omega} \setminus \{\emptyset\}$ that can be covered by finitely many infinite branches (i.e. elements of $2^{\mathbf{N}}$). To see that (H, τ_1) is Frechet, let $Y \subseteq H$ be such that $\emptyset \in \overline{Y}$. It is easy to check that Y must contain an infinite antichain D. Let x_n be an enumeration of D. Then x_n converges to $\emptyset$.

Let now A be the irrational points of $2^{\mathbf{N}}$ and denote τ_A by τ_2. The space (H, τ_2) has the property of not being embeddable into $C_p(K)$ for any compact metric space K (if it was, then one easily shows that the set of irrational points would be F_σ). A space with the same property was given by R. Pol [18]. However, as we will see, (H, τ_2) is embeddable into $C_p(\mathbf{N}^{\mathbf{N}})$ and moreover its pointwise closure is a subset of the collection of Baire class 1 functions on $\mathbf{N}^{\mathbf{N}}$. $\square$

6 Embedding a countable analytic space into $C_p(\mathbf{N}^{\mathbf{N}})$

It is not an accident that many examples of countable analytic spaces are variations of the space (G, τ_A) presented in §5. In fact this is a quite universal construction. To see this consider an analytic T_0 topology τ on a countable set Y. Let $f : \mathbf{N}^{\mathbf{N}} \to 2^Y$ be a continuous map whose range is equal to τ. For $y \in Y$, let

$$y^* = \{x \in \mathbf{N}^{\mathbf{N}} : \ y \in f(x)\}$$

Then $Y^* = \{y^* : \ y \in Y\}$ is a countable family of clopen subsets of $\mathbf{N}^{\mathbf{N}}$. Let τ^* be the topology on Y^* generated by subbasis

$$x^+ = \{y^* \in Y^* : \ x \in y^*\}, \ \ (x \in \mathbf{N}^{\mathbf{N}})$$

It is clear that (Y, τ) is homeomorphic to (Y^*, τ^*) via the mapping $y \mapsto y^*$.

Suppose now that (Y, τ) is a regular T_2 topological space, then the family

$$\tau \cap \tau^c = \{U \subseteq Y : \ U, U^c \in \tau\}$$

of all τ-clopen subsets of Y is also analytic. So, let $f : \mathbf{N}^{\mathbf{N}} \to 2^Y$ be a continuous maps whose range is equal to $\tau \cap \tau^c$. Let Y^* be as before but let τ^* be now the topology on Y^* generated by subbasic clopen sets of the form

$$x^+ = \{y^* \in Y^* : x \in y^*\} \text{ and } x^- = \{y^* \in Y^* : x \notin y^*\}$$

where $x \in \mathbf{N}^{\mathbf{N}}$. Thus, if we identify sets with their characteristic functions we get a copy (Y^*, τ^*) of our regular space (X, τ) inside the function space $C_p(\mathbf{N}^{\mathbf{N}})$ where the p stands for the topology of pointwise convergence. If we denote by $C_{p^+}(\mathbf{N}^{\mathbf{N}})$ the weaker topology of $C(\mathbf{N}^{\mathbf{N}})$ with subbasic open sets of the form

$$B(x, q) = \{h \in C(\mathbf{N}^{\mathbf{N}}) : \ h(x) > q\} \tag{2}$$

where $x \in \mathbf{N}^{\mathbf{N}}$ and $q \in \mathbf{R}$, we get the following representation result.

Proposition 6.1. *Let (X, τ) be a countable T_0 space.*

(i) τ is analytic iff X is homeomorphic to a subspace of $C_{p^+}(\mathbf{N}^{\mathbf{N}})$. If moreover the space is regular and T_2, then it is actually homeomorphic to a subspace of $C_p(\mathbf{N}^{\mathbf{N}})$.

(ii) X is homeomorphic to a subspace of $C_{p^+}(K, \{0,1\})$ for K compact metric iff X has a compact subbase.

(iii) X is homeomorphic to a subspace of $C_p(K, \{0,1\})$ for K compact metric iff X has a compact subbase of clopen sets.

Proof: Let $X \subseteq C_p(\mathbf{N}^{\mathbf{N}})$ be a countable subspace. Let $B(x,q)$ as in (2) and observe that the map $(x,q) \mapsto B(x,q) \cap X$ from $\mathbf{N}^{\mathbf{N}} \times \mathbf{R}$ into 2^X is Borel and its range is a subbase for X. Then apply 3.2. This shows ($\Leftarrow$) in (i) and the other direction was proved before. To see (iii) let $X \subseteq C_p(K, \{0,1\})$ be a countable subspace with K compact metric. For each $a \in K$, let $f(a) = \{x \in X : x(a) = 1\}$. Then f is a continuous map from K into 2^X. Let $K^* = \{f(a) : a \in K\} \cup \{X \setminus f(a) : a \in K\}$. The K^* is a compact subbase of clopen sets of X. Conversely suppose that X has a compact subbase K of clopen sets. Then for each $x \in X$ let $x^+ = \{O \in K : x \in O\}$. Each x^+ is a clopen subset of K. Identifying x^+ with its characteristic function we get an embedding $x \mapsto x^+$ from X into $C_p(K, \{0,1\})$. The proof of (ii) is similar. $\square$

It is clear that in order to define an embedding from (Y, τ) into $C_{p^+}(\mathbf{N}^{\mathbf{N}})$ (resp. into $C_p(\mathbf{N}^{\mathbf{N}})$) one needs to start from a continuous $f : \mathbf{N}^{\mathbf{N}} \to 2^Y$ whose range is only a subbase of open sets (resp. clopen sets). This gives an advantage of choosing good embeddings $y \mapsto y^*$. For example, in the case of embeddings into $C_p(\mathbf{N}^{\mathbf{N}})$ one is tempted to take the closure of Y^* inside the Tychonov cube $\{0,1\}^{\mathbf{N}^{\mathbf{N}}}$ and obtain a natural compactification of (Y, τ). It is clear that different choices of subbasis of (Y, τ) may result in a quite different compactifications. There is a beautiful result of Rosenthal [19] about taking pointwise closure of a bounded set of continuous functions on $\mathbf{N}^{\mathbf{N}}$: The pointwise closure either embeds $\beta\mathbf{N}$ and therefore has size bigger than the continuum or it is included in $\mathcal{B}_1(\mathbf{N}^{\mathbf{N}})$, the space of all Baire class-1 functions on $\mathbf{N}^{\mathbf{N}}$ with the pointwise topology. Today compact subspaces of $\mathcal{B}_1(\mathbf{N}^{\mathbf{N}})$ are called *Rosenthal compacta*. So, in our situation, it is appropriate to call the closure of Y^* a *Rosenthal compactification* of (Y, τ) in case it is included in $\mathcal{B}_1(\mathbf{N}^{\mathbf{N}})$. A famous result of Bourgain-Fremlin-Talagrand [3] can now be stated as follows.

Theorem 6.2. *If a regular countable analytic space (Y, τ) has a Rosenthal compactification, then (Y, τ) is a Fréchet space.*

The role of analytic topologies on countable sets in analyzing the class of Rosenthal compacta is crucial as the following reformulation of a result of Godefroy [10] shows.

Theorem 6.3. *The following conditions are equivalent for every separable compact space K:*
(1) K is embeddable into the first Baire class.
(2) The uniformity K induces on any of its countable dense subsets is analytic.

Rosenthal's dichotomy can now be restated as a result about analytic topologies over countable sets as follows (see also [10, p. 305])

Theorem 6.4. *The following three conditions are equivalent for a regular countable space (Y, τ):*

(1) (Y,τ) *has a Rosenthal compactification in* $\mathcal{B}_1(\mathbf{N}^{\mathbf{N}},\{0,1\})$.

(2) There is an analytic subbasis $\mathcal{B}\subseteq\tau$ *of clopen sets such that there is no infinite* $Z\subseteq Y$ *such that* $\mathcal{B}|Z=\mathcal{P}(Z)$.

(3) There is an analytic subbasis $\mathcal{B}\subseteq\tau$ *of clopen sets such that for every infinite* $Z\subseteq Y$ *there is an infinite* $Z_\infty\subseteq Z$ *such that for every* $B\in\mathcal{B}$ *either* $Z_\infty\setminus B$ *or* $Z_\infty\cap B$ *is finite.*

Proof: Suppose that Y has a Rosenthal compactification in $\mathcal{B}_1(\mathbf{N}^{\mathbf{N}},\{0,1\})$. We may assume that $Y\subseteq C_p(\mathbf{N}^{\mathbf{N}},\{0,1\})$ (by increasing a bit the topology of $\mathbf{N}^{\mathbf{N}}$ if necessary) and that the pointwise closure of Y is a subset of the first Baire class of $\mathbf{N}^{\mathbf{N}}$. Let

$$\mathcal{B}=\{\{y\in Y:\ y(a)=0\}:\ a\in\mathbf{N}^{\mathbf{N}}\}$$

It is clear that $\mathcal{B}$ is a subbasis of Y consisting of clopen sets. To see that $\mathcal{B}$ satisfies (2) assume by way to a contradiction, that $\mathcal{B}|Z=\mathcal{P}(Z)$ for some infinite $Z\subseteq Y$. Since $K=\overline{Y}$ is compact, for every ultrafilter $\mathcal{U}$ on Z there is a unique point $y(\mathcal{U})\in K$ such that $\{G\cap Z:\ G \text{ open}, y(\mathcal{U})\in G\}$ is included in $\mathcal{U}$. Note that by our assumption $\mathcal{B}|Z=\mathcal{P}(Z)$, $y(\mathcal{U})\neq y(\mathcal{V})$ whenever $\mathcal{U}\neq\mathcal{V}$. It follows that K has cardinality bigger than the continuum, a contradiction.

Assume now that (2) holds and fix a continuous map $f:\mathbf{N}^{\mathbf{N}}\to\mathcal{P}(Y)$ such that $\mathcal{B}=range(f)$ and consider the copy $Y^*=\{y^*:\ y\in Y\}$ defined in the proof of 6.1 (i), i.e. y^* is the characteristic function of the clopen set $\{a\in\mathbf{N}^{\mathbf{N}}:\ \ y\in f(a)\}$. We claim that the pointwise closure K of Y^* in $\{0,1\}^{\mathbf{N}^{\mathbf{N}}}$ is included in the first Baire class of $\mathbf{N}^{\mathbf{N}}$. Otherwise, using Rosenthal's dychotomy theorem (see [19]) there is a 1-1 mapping $x\mapsto a_x$ from $2^{\mathbf{N}}$ into $\mathbf{N}^{\mathbf{N}}$ and a subsequence $\{y_n^*:\ n\in\mathbf{N}\}$ such that for every n,

$$y_n^*(a_x)=1 \ \text{ iff } \ x(n)=1$$

Let $Z=\{y_n:\ n\in\mathbf{N}\}$. To get the desired contradiction we will show that for every $A\subseteq\mathbf{N}$ there is $B\in\mathcal{B}$ such that $B\cap Z=\{y_n:\ n\in A\}$. To see this let $x\in 2^{\mathbf{N}}$ be the characteristic function of A and $B=f(a_x)$. It is easy to check that B works.

To show that (1) implies (3) it suffices to show that the subbasis $\mathcal{B}$ constructed in the course of the proof of (1) $\Rightarrow$ (2) also satisfies the stronger condition (3). This follow from Rosenthal's theorem according to which every infinite sequence $Z=\{z_n\}_{n=0}^\infty$ contains a converging subsequence $Z_\infty=\{z_{n_k}\}_{k=0}^\infty$. The proof that (3) implies (1) follows from the fact that (3) is stronger than (2) (just observe that if Z_∞ is split into two disjoint infinite sets, then none of the pieces can be in $\mathcal{B}|Z$).

$\square$

Remark 6.5. Note that the space (H,τ_2) considered in example 5.6 satisfies condition (3) of theorem 6.4 and therefore admits a Rosenthal compactification and, in particular, it is Fréchet. In fact, let $\mathcal{B}$ be the collection of all $\{s\}$ with $s\in 2^{<\omega}$ together with the subsets A of $2^{<\omega}$ such that $2^{<\omega}\setminus A$ can be covered by finitely many irrational branches. Then $\mathcal{B}$ is a base for H. Let $Z\subseteq H$ be infinite. Then there are two cases: either Z contains an infinite chain Z_∞ or it contains an infinite antichain Z_∞. For $B\in\mathcal{B}$ a neighbourhood of $\emptyset$, these two cases correspond to the two alternatives given in (3).

Another result worth mention is the following fact closely related to a result called 'Szlenk's theorem' (see [23]) by Pol [18]

Theorem 6.6. *The following are equivalent for an countable analytic space (Y, τ) and a point $o \in Y$.*

1. *Y is Fréchet at o and whenever $\{y_{m,n}\}$ is a double sequence of elements of Y such that $lim_n y_{m,n} = o$ for each $m \in \mathbf{N}$, then for each m we can choose $n(m) \in \mathbf{N}$ such that $\{y_{m,n(m)}\}$ converges to o.*

2. *o has a countable neighbourhood base in Y.*

Proof: To prove the non trivial implication (1) $\Rightarrow$ (2), let $\{y_n : n \in \mathbf{N}\}$ be a fixed enumeration of $Y \setminus \{o\}$. Let

$$\begin{aligned} A &= \{a \subseteq \mathbf{N} : \{y_n : \ n \in a\} \text{ does not accumulate to } o\} \\ B &= \{b \subseteq \mathbf{N} : \{y_n : \ n \in b\} \text{ converges to } o\} \end{aligned}$$

Then A and B are two orthogonal families of subsets of $\mathbf{N}$ and, since τ is analytic, it follows easily that A is analytic as a subset of the Cantor set $2^{\mathbf{N}}$. Note that (2) reduces to the fact that A, an ideal of subsets of $\mathbf{N}$, is countably generated. Note also that the assumption that Y is Frechet space at o reduces to the fact that every $a \subseteq \mathbf{N}$ which has a finite intersection with every member of B must belong to A, or in the terminology of [24], that $B^{\perp} = A$. So (2) is equivalent to the statement that A is countably generated in $B^{\perp}$. By theorem 3 of [24] if this fails there must be a nonempty family T of finite subsets of $\mathbf{N}$ closed under taking initial segments such that

(a) $b_s = \{n \in \mathbf{N} : \ n > \max(s) \text{ and } s \cup \{n\} \in T\}$ belongs to B for every $s \in T$.

(b) Every $a \subseteq \mathbf{N}$ with the property that $a \cap \{0, \cdots, n-1\} \in T$ for all $n \in \mathbf{N}$ must belong to A.

Applying (1) to the family $\{b_s : \ s \in T\}$ of sequences converging to o we get for each $s \in T$ a point $i_s \in b_s$ such that $b = \{i_s : \ s \in T\}$ converges to o, i.e. belongs to B. However, note that by (b) the sequence $\sigma : \mathbf{N} \to b$ defined recursively by $\sigma(n) = i_{\sigma|n}$ has the property that its range is an infinite subset of b which belongs to A, a contradiction. $\square$

7 Ideals of nowhere dense sets.

Given a topology τ over X, we will denote by $NWD(\tau)$ the collection of τ-nowhere dense sets, i.e. those subsets $A \subseteq X$ such that $cl_\tau(A)$ has empty interior. In this section we address the question of representing a given ideal over X as the nowhere dense sets with respect to a topology over X. This problem has been studied in [5]. Let $\mathcal{I}$ be an ideal over X containing all singletons. Then the dual filter (together with $\emptyset$) is a T_1 (but not Hausdorff) topology such that its nowhere dense sets are exactly the sets in $\mathcal{I}$. Here we are interested in the following question: given a Borel (analytic) ideal $\mathcal{I}$ over X, what are the possible topologies τ such that $\mathcal{I} = NWD(\tau)$? For example, it is known that there is no Hausdorff topology τ such that $NWD(\tau) = \text{FIN}$ (see [5]). We will see that this result extends to F_σ ideals.

Let $\mathcal{I}$ and $\mathcal{J}$ be two ideal on $\mathbf{N}$. We say that they are equivalent, denoted by $\mathcal{I} \equiv \mathcal{J}$, if there is a bijection from $\mathbf{N}$ onto $\mathbf{N}$ such that $A \in \mathcal{I}$ if and only if $f^{-1}(A) \in \mathcal{J}$. There are two orders to compare ideals of subsets of $\mathbf{N}$ (or any countable set) which has been very successfully used to study the structural properties of definable ideals. The first one, denoted by $\leq_{TK}$, is called the relation of *Tukey reducibility*: $\mathcal{I} \leq_{TK} \mathcal{J}$ if there is a monotone (with respect to $\subseteq$) map $f : \mathcal{J} \to \mathcal{I}$ which maps $\mathcal{J}$ onto a cofinal subset of $\mathcal{I}$, or equivalently, if there is a map $g : \mathcal{I} \to \mathcal{J}$ such that

$\{A \in \mathcal{I} : f(A) \subseteq B\}$ is bounded in $\mathcal{I}$ for every $B \in \mathcal{J}$. The map g is called a *Tukey map* from $\mathcal{I}$ into $\mathcal{J}$. It is not hard to see that this is equivalent to saying that there is a Moore-Smith convergent map from $\mathcal{J}$ into $\mathcal{I}$. We say that two ideals $\mathcal{I}$ and $\mathcal{J}$ are *Tukey equivalent*, denoted by $\mathcal{I} \equiv_{TK} \mathcal{J}$, if $\mathcal{I} \leq_{TK} \mathcal{J}$ and $\mathcal{J} \leq_{TK} \mathcal{I}$. The second order, denoted by $\leq_{RB}$, is defined as follows: $\mathcal{I} \leq_{RB} \mathcal{J}$ if there is a finite-to-one map (called a *Rudin-Blass reduction*) $h : \mathbf{N} \to \mathbf{N}$ such that $h^{-1}(A) \in \mathcal{J}$ iff $A \in \mathcal{I}$. Mathias [15] has shown that every analytic ideal $\mathcal{I}$ is Rudin-Blass reducible to FIN, and this was later extended by Jalali-Naini and Talagrand who showed that the relation FIN $\leq_{RB} \mathcal{I}$ is a characterization of the class of Baire-measurable ideals on $\mathbf{N}$.

We will start by looking at $NWD(\tau)$ for τ an Alexandroff topology (i.e. by 2.2 a topology which is closed as a subset of the Cantor cube).

Theorem 7.1. *Let $\mathcal{I}$ be an ideal over a countable set X. Then $\mathcal{I} = NWD(\tau)$ for some Alexandroff topology τ over X if, and only if $\mathcal{I}$ is equivalent to a free sum of ideals belonging to the following family: principal ideals, FIN, FIN $\times\, \emptyset$ and $NWD(\mathbf{Q})$.*

We start by showing that all ideals belonging to the family mentioned in theorem 7.1 are representable by an Alexandroff topology.

Proposition 7.2. *If $\mathcal{I}$ is either a principal ideal , FIN, FIN $\times\, \emptyset$ or $NWD(\mathbf{Q})$, then there is a T_0 Alexandroff topology τ such that $\mathcal{I} = NWD(\tau)$.*

Proof: We will define for each case a partial order $\leq_\tau$ and the topology will be given by 2.1.

For a principal ideal $\mathcal{P}(A)$, let $\leq_\tau$ be defined by $x <_\tau y$ for all $x \in A$ and $y \notin A$. For FIN, let $\leq_\tau$ be the usual order over ω.

For FIN $\times\, \emptyset$, let $\leq_\tau$ be defined over $\omega \times \omega$ as follows: $(n, m) <_\tau (n', m')$ if $n < n'$ and $(n, m) <_\tau (n, m')$ if $m' < m$, so the order of $\{n\} \times \omega$ is the reversed order of ω. In other words, we have put a copy of ω^* for each element of ω. This is a total order without a maximal point, hence a set is nowhere dense iff it is bounded. From this the result easily follows.

For $NWD(\mathbf{Q})$, let τ be the smallest topology that makes clopen all cones w.r.t. the usual extension order over $X = \omega^{<\omega} - \{\emptyset\}$ (i.e. that makes clopen the sets $\{t : s \prec t\}$ for all $s \in \omega^{<\omega} - \{\emptyset\}$). Then (X, τ) is homeomorphic to $\mathbf{Q}$. On the other hand, the identity map witnesses that $NWD(X, \tau) \equiv NWD(\omega^{<\omega} - \{\emptyset\})$ (where $\omega^{<\omega} - \{\emptyset\}$ is given the Alexandroff topology of the usual extension order). $\square$

The next proposition takes care of some cases in the *only if* part of 7.1.

Proposition 7.3. *Let $\leq$ be a quasi-order over X which is up-directed. One of the following holds:*

(i) $NWD(X)$ is principal.

(ii) $NWD(X)$ is a trivial variation of FIN (i.e. there is $B \subseteq X$ such that $A \in NWD(X)$ if, and only if $A \cap B$ is finite. Thus $NWD(X)$ is the free sum of a principal ideal and FIN).

(iii) $NWD(X) \equiv$ FIN $\times\, \emptyset$

Proof: Let M be the set of all maximal elements of X, then $NWD(X) = \mathcal{P}(X - M)$ and hence (i) holds. So we assume that X has no maximal elements. Let (x_n) be a cofinal sequence linearly ordered. Let $A_n = \{x \in X : x \leq x_n\}$, note that $A \in NWD(X)$ if, and only if there is n such that $A \subseteq A_n$. It is known that this condition implies that either (ii) or (iii) hold (see [12]). In fact, we consider two cases.

Case 1: There is N such that for all $n \geq N$, $A_{n+1} - A_n$ is finite. We will show that (ii) holds. Let $B = X - A_N$ and $A \in NWD(X)$. Let n be such that $A \subseteq A_n$. If $n \leq N$ then $A \cap B$ is empty, so we assume that $n > N$. We have that $A \cap B = A \cap (\bigcup_{i=N}^{n} A_{i+1} - A_i)$ and therefore $A \cap B$ is finite. On the other hand, if $A \cap B$ is finite, then it is clear that there is n such that $A \subseteq A_n$.

Case 2: For infinitely many n, $A_{n+1} - A_n$ is infinite. By passing to a subsequence we can assume that for all n, $A_{n+1} - A_n$ is infinite. Let $\{x_m^{n+1}\}_m$ be an enumeration of $A_{n+1} - A_n$ and $\{x_m^0\}_m$ be an enumeration of A_0. Notice that $X = \{x_m^n : n, m \geq 0\}$. Let $f : X \to \omega \times \omega$ be defined by $f(x_m^n) = (n, m)$. Then $A \in NWD(X)$ if, and only if $f[A] \in \mathrm{FIN} \times \emptyset$. $\square$

Proposition 7.4. *Let $\leq$ be an everywhere branching quasi-order over X (i.e. for every x there are y, z such that $x \leq y$, $x \leq z$ and y and z are incompatible). Then $NWD(X) \equiv NWD(\omega^{<\omega} - \{\emptyset\})$.*

Proof: It is not difficult to find an isomorphic copy T of $\omega^{<\omega} - \{\emptyset\}$ inside X which is cofinal in X (by induction, using the fact that every element of X has infinitely many pairwise incompatible successors). We can also assume w.l.o.g. that T is isomorphic to the collection of non-empty sequences of even length. The idea to define the isomorphism between $\omega^{<\omega} - \{\emptyset\}$ and X is to fill the collection of sequences of odd length with $X \setminus T$. For each $n \geq 1$, let B_n be the collection of all $x \in X \setminus T$ such that $x \leq t$ for some $t \in T$ with length $2n$. Notice that $B_n \subseteq B_{n+1}$ and the union of all B_n is $X \setminus T$ as T is cofinal in X. We can also assume w.l.o.g. that B_1 and $B_{n+1} \setminus B_n$ are infinite (if not, then substitute T by its sequences of length $4n$). Define f from X into $\omega^{<\omega} - \{\emptyset\}$ as the identity on T, elements of B_1 are mapped onto the sequences of length 1 and elements of $B_{n+1} \setminus B_n$ are mapped onto the sequences of length $2n + 1$. $\square$

Proof of 7.1: Note that the ideal of nowhere dense sets of a free sum of topologies is equivalent to the free sum of the corresponding ideals. Also, the free sum of Alexandroff topologies is represented by the free sum of the corresponding partial orders. From this and 7.2 the *if* part of the theorem follows.

Let $\leq$ be a quasi-order over X. Let O be an open dense subset of X. We first show that we can restrict the question to $NWD(O)$. We consider two cases: (a) Suppose that every set in $NWD(O)$ is finite, then we have that $A \in NWD(X)$ iff $A \cap O$ is finite. Hence $NWD(X)$ is a trivial variation of FIN. (b) Suppose $F \in NWD(O)$ is infinite and fix a bijection g between $F \cup (X \setminus O)$ and F. Define a bijection from X onto O by letting $f(x) = x$ for $x \notin F \cup (X \setminus O)$ and $f(x) = g(x)$ for $x \in F \cup (X \setminus O)$. It is easy to check that f is an isomorphism between $NWD(X)$ and $NWD(O)$.

Consider the following subsets of X

$$P_0 = \{x \in X : \ N_x \text{ is up-directed}\} \qquad P_1 = \{x \in X : \ N_x \cap P_0 = \emptyset\}$$

where $N_x = \{y \in X : \ x \leq y\}$. It is easy to check that P_0 and P_1 are open sets and $P_0 \cup P_1$ is dense in X. From the remark above we can assume that $X = P_0 \cup P_1$. Since P_0 and P_1 are open and disjoint, then $NWD(X) = NWD(P_0) \bigoplus NWD(P_1)$. Now, let $\{D_n\}$ be the collection of all maximal up-directed subsets of P_0. Then $NWD(P_0) = \bigoplus_n NWD(D_n)$. It is obvious that P_1 is everywhere branching. Now the conclusion follows from 7.3 and 7.4. $\square$

Now we will address the question of when a given ideal is representable by a Hausdorff topology. First of all, let us observe that if (X, τ) is scattered, then by a simple induction on the Cantor-Bendixon rank of X it is easy to check that $NWD(\tau) = \mathcal{P}(X^{(1)})$, so $NWD(\tau)$ is principal. Also observe that by 4.1 a given $G_{\delta\sigma}$ Hausdorff topology τ can have only finitely many limit points, thus in this case $NWD(\tau)$ is also principal. So in order to represent non principal ideals with Hausdorff topologies we must look for topologies which are as least $\mathbf{\Pi}_3^0$ and not scattered.

We start by showing that if τ is a Hausdorff topology without isolated points then $NWD(\tau)$ is at least as complex as $\emptyset \times \mathrm{FIN}$ in the Tukey sense.

Theorem 7.5. *Let (X, τ) be a Hausdorff space without isolated points. There exists $F : NWD(\tau) \to \mathbf{N}^{\mathbf{N}}$ monotone, continuous and with cofinal range. In other words, $\emptyset \times FIN \leq_{TK} NWD(\tau)$ and moreover the map witnessing this is continuous.*

Proof: Let $\{U_n\}$ be a pairwise disjoint family of non empty open sets. Let $U_n = \{x_n(i)\}_{i=1}^{\infty}$ and $n \in \omega$. Define F as follows: for $S \in NWD(\tau)$, put

$$F(S)(n) = \max\{k : \{x_n(i)\}_{i=1}^{k} \subseteq S\}$$

It is clear that F is continuous and also that $F(S)(n) \leq F(S')(n)$, if $S \subseteq S'$, i.e. F is monotone. To see that F is onto, let $h \in \mathbf{N}^{\mathbf{N}}$ and $S = \{x_n(i) : n \in \omega, i \leq h(n)\}$. Assuming that $S \in NWD(\tau)$ it is clear that $F(S) = h$. To show that $S \in NWD(\tau)$ we observe that for every $n \in \omega$, $\overline{S} \cap U_n = S \cap U_n$ is finite. So if $V \subseteq \overline{S}$ with V non empty and open then there must be an n such that $V \cap U_n$ is non empty, therefore there is an open subset of U_n contained in S which is a contradiction since every non empty open set is infinite. $\square$

We shall now show that in studying $NWD(\tau)$ for T_2 topologies τ over countable sets we may restrict ourselves to topologies that extend the topology of the rationals.

Definition 7.6. *A π-base of a topological space (X, τ) is any family $\mathcal{P} \subseteq \tau \setminus \{\emptyset\}$ with the property that for every nonempty $U \in \tau$ there is $V \in \mathcal{P}$ such that $V \subseteq U$.*

The relevance of this notion here is that a π-base $\mathcal{P}$ of (X, τ) uniquely determine the family $NWD(\tau)$ as $N \in NWD(\tau)$ iff for all $U \in \mathcal{P}$ there is $V \in \mathcal{P}$ such that $V \subseteq U$ and $V \cap N = \emptyset$.

Lemma 7.7. *Let G be a regular open subset of some space (X, τ) and let τ^* be the topology on X generated by $\tau \cup \{X \setminus G\}$. Then $\tau \setminus \{\emptyset\}$ is a π-base of τ^*.*

Proof: A typical nonempty open set of τ^* has the form $V \setminus G$ for some $V \in \tau$. Since G is regular open, $V \setminus G \neq \emptyset$ implies $V \setminus \overline{G} \neq \emptyset$, so $V \setminus \overline{G}$ is a nonempty τ-open set which refines $V \setminus G$. $\square$

Theorem 7.8. *For every T_2 topology τ on some countable set X there is a topology $\tau^* \supseteq \tau$ on X such that*

(a) τ^* is generated by τ together with some countable collection of subsets of X.

(b) $\tau \setminus \{\emptyset\}$ is a π-base of τ^*, so in particular, $NWD(\tau^*) = NWD(\tau)$.

(c) There is a continuous injection $f : (X, \tau^*) \to \mathbf{Q}$.

Proof: Fix an enumeration $\{x_n\}$ of X and using 7.7 build sequences $\tau = \tau_0 \subseteq \tau_1 \subseteq \cdots$ of topologies on X and $\{G_n\}$ of subsets of X such that

(i) G_n is regular-open in τ_n,

(ii) $x_n \in G_n$ and $x_i \notin cl_{\tau_n}(G_n)$ for $i < n$,

(iii) τ_{n+1} is generated by $\tau_n \cup \{X \setminus G_n\}$

Let τ^* be the topology generated by $\bigcup_{n=0}^{\infty} \tau_n$. Taking $[\omega]^{<\omega}$ with the subspace topology induced from the Cantor set as our copy of $\mathbf{Q}$, define $f : X \to [\omega]^{<\omega}$ by $f(x) = \{n : x \in G_n\}$. Clearly f is 1-1 and τ^*-continuous as subbasic clopen sets of $[\omega]^{<\omega}$ are sets of the form $\{t : t \text{ end extends } s\}$ whose preimage under f is equal to $\bigcap_{n\in s} G_n \cap \bigcap_{n\notin s, n<max(s)} G_n^c$, a set which is clopen in τ^*. $\square$

Remark 7.9. (i) Suppose that (X, τ) is Hausdorff and without isolated points. Is there $F : NWD(\tau) \to NWD(\mathbf{Q})$ monotone, continuous and with cofinal range?

(ii) Notice that $(NWD(\mathbf{Q}))^\omega \equiv_{TK} NWD(\mathbf{Q})$. So in general, let τ be a T_2 topology without isolated points, is it true that $(NWD(X))^\omega \equiv_{TK} NWD(X)$?

8 Complexities of ideals of nowhere dense sets

In this section we will address the question of the complexity of $NWD(\tau)$. Let us start by calculating the upper bound of the projective complexity of $NWD(\tau)$ when τ is analytic.

$$A \in NWD(\tau) \quad \text{if, and only if} \quad \forall V \in \tau \setminus \{\emptyset\}\ \exists W \in \tau \setminus \{\emptyset\}\ (W \subseteq V \ \&\ W \cap A = \emptyset\ )$$

From this it follows that $NWD(\tau)$ is $\mathbf{\Pi}^1_2$. If τ is second countable then by a direct calculation it is easy to see that $NWD(\tau)$ is $\mathbf{\Pi}^0_3$. We state this observation for later reference.

Proposition 8.1. *Let τ be a second countable topology, then $NWD(\tau)$ is $\mathbf{\Pi}^0_3$.* $\square$

From 7.5 we know that if τ is Hausdorff without isolated points then $NWD(\tau)$ is not F_σ. We will show next a stronger result.

Theorem 8.2. *Let τ be a Hausdorff topology over a countable set X without isolated points and $\mathcal{I}$ be a proper F_σ ideal over X. Then $NWD(\tau) \not\subseteq \mathcal{I}$. In particular, $NWD(\tau)$ is not F_σ.*

Proof: Let $\mathcal{I}$ be a proper F_σ ideal over X. We can assume w.l.o.g. that $\mathcal{I} = \bigcup_n F_n$ with each F_n closed hereditary and $F_n \subseteq F_{n+1}$. We consider two cases:

Case 1: $\tau \cap \mathcal{I} = \{\emptyset\}$. Since τ is Hausdorff, let $\{V_n\}$ be an infinite family of nonempty pairwise disjoint open sets. By assumption $V_n \notin \mathcal{I}$. Since F_n is closed we have that for a given $A \subseteq X$ if every finite subset of A belongs to F_n, then $A \in F_n$. Then for each n, let K_n be a finite subset of V_n such that $K_n \notin F_n$. Let $A = \bigcup_n K_n$, since each F_n is hereditary then $A \notin F_n$, i.e. $A \notin \mathcal{I}$. On the other hand, $A \in NWD(\tau)$ because every finite set is τ-nowhere dense and $A \cap V_n$ is finite. $\square$

Case 2: $\tau \cap \mathcal{I} \neq \{\emptyset\}$. Suppose, towards a contradiction, that $NWD(\tau) \subseteq \mathcal{I}$. Let $V = \bigcup\{O : O \in \tau \cap \mathcal{I}\}$, then $NWD(\tau|V) = NWD(\tau) \cap \mathcal{P}(V)$ and hence we can assume that $X = V$. (In fact, using Case 1 it is easy to check that V is dense in X, but this will not be used). Our assumption is then that for all $x \in X$ there is an open set O such that $x \in O \in \mathcal{I}$. Notice that if O is an open set in $\mathcal{I}$, then $\overline{O} \in \mathcal{I}$ (since $\overline{O} \setminus O$ is nowhere dense).

We will construct two sequences $\{K_n\}$ of finite sets and $\{U_n\}$ of open sets such that (a) $K_n \subseteq U_n$, (b) $K_n \notin F_n$, (c) $U_n \in \mathcal{I}$ and (d) $U_n \cap U_m = \emptyset$ for $n \neq m$. Since $X \notin \mathcal{I}$, pick a finite set $K_0 \notin F_0$

and let $U_0 \in \mathcal{I}$ such that $K_0 \subseteq U_0$. Suppose that U_j and K_j have been constructed for $j < n$. Let $D = \bigcup_{j=0}^{n-1} \overline{U_j}$, then $D \in \mathcal{I}$. Therefore $X \setminus D \notin \mathcal{I}$ and thus there is a finite set $K_n \subseteq X \setminus D$ such that $K_n \notin F_n$. Let U_n be an open set such that $K_n \subseteq U_n \in \mathcal{I}$ and $U_n \cap D = \emptyset$. Let $A = \bigcup_n K_n$. As we did in Case 1, it is easy to show that A is nowhere dense and $A \notin \mathcal{I}$. $\Box$

If τ is a Hausdorff topology without isolated points and it is moreover analytic then we can conclude more, namely, that $NWD(\tau)$ is at least $\mathbf{\Pi}^0_3$.

Theorem 8.3. *Let τ be an analytic Hausdorff topology without isolated points, then $\emptyset \times FIN \leq_{RB} NWD(\tau)$. If τ is in addition second countable, then $NWD(\tau)$ is $\mathbf{\Pi}^0_3$-complete.*

Proof: Let τ be an analytic T_2 topology without isolated points. Fix a maximal cellular family $\{U_n\}$ of open sets. First we argue that it suffices to show that FIN $\leq_{RB} NWD(\tau|U_n)$ for every n. In fact, suppose $h_n : U_n \to \mathbf{N}$ is a Rudin-Blass reduction of $NWD(\tau|U_n)$ to FIN. Let $F = X - \bigcup_n U_n$, notice that F is closed nowhere dense. Define $h : X \to \mathbf{N} \times \mathbf{N}$ by $h(x) = (n, h_n(x))$ if $x \in U_n$ and $h(x) = (0, 0)$ if $x \in F$. Then $h^{-1}(A) \subseteq \bigcup_n h_n^{-1}(\{i : (n, i) \in A\}) \cup F$. Thus $h^{-1}(A) \in NWD(\tau)$ if, and only if $A \in \emptyset \times$ FIN.

Fix a non empty open set U and let $\{x_i\}$ be an enumeration of U. By the theorem of Mathias, already used before, for every i there is a collection $\{F^i_n : i \leq n, n \in \mathbf{N}\}$ of pairwise disjoint finite subsets of $U - \{x_i\}$ such that $\bigcup_{n \in A} F^i_n$ accumulates to x_i for each infinite $A \subseteq \mathbf{N}$. By a standard diagonalization process we can find infinite sets $A_i \subseteq \mathbf{N}$, for each $i \in \mathbf{N}$, such that if $i \neq j$, then $F^i_n \cap F^j_m = \emptyset$ for all $n \in A_i$ and all $m \in A_j$. In other words, we can assume that $\{F^i_n : i \leq n, i, n \in \mathbf{N}\}$ is pairwise disjoint. Also, we can assume that $U = \bigcup_{(i,n)} F^i_n$ (if a point x_k does not belong to any F^i_n then we add x_k to F^{k+1}_{k+1}). Define $h : U \to \mathbf{N}$ by

$$h(x) = m, \text{ if } x \in F^i_m$$

Then $h^{-1}(m) = F^1_m \cup \cdots \cup F^m_m$, thus h is finite-to-one. To see that h is a Rudin-Blass reduction, let $A \subseteq \mathbf{N}$ be infinite, then $\{n : F^i_n \subseteq h^{-1}(A)\}$ is infinite for each i. Therefore $x_i \in \overline{h^{-1}(A)}$ and thus $U \subseteq \overline{h^{-1}(A)}$. This shows that FIN $\leq_{RB} NWD(\tau|U)$.

The last claim follows from 8.1 and the fact that $\emptyset \times$ FIN $\leq_W NWD(\tau)$. $\Box$

Now we will show that $NWD(\tau)$ can not be p-ideal when τ is analytic. Moreover, we will also show that $NWD(\tau)$ is not included in any proper analytic p-ideal. This result is related to [17, Problem 256], which asks whether $NWD(\mathbf{Q})$ can be extended to a p-ideal. Recall that an ideal $\mathcal{I}$ over a countable set X is called a *p-ideal* if for every sequence $A_n \in \mathcal{I}$, there is a set $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n.

We will need the following general fact which is interesting by itself.

Lemma 8.4. *Let τ be a Hausdorff topology over X without isolated points. If $NWD(\tau)$ is a p-ideal, then there is a dense open set $U \subseteq X$ such that*

$$NWD(\tau|U) = \{A \subseteq U : \ A \text{ is closed discrete in } U\} \tag{3}$$

Proof: Let us say that a point $x \in X$ is *near point* if there is a nowhere dense set A such that $x \notin A$ and $x \in \overline{A}$. Let Z be the collection of near points of X. Fix for every $x \in Z$ a nowhere dense set A_x such that $x \in \overline{A_x \setminus \{x\}}$. Since $NWD(\tau)$ was assumed to be a p-ideal, there is a nowhere dense set B such that $A_x \setminus B$ is finite for all $x \in Z$. Notice that $Z \subseteq \overline{B}$. Let $U = X \setminus \overline{B}$. To show that U

works first observe that $NWD(\tau|U) = NWD(\tau) \cap \mathcal{P}(U)$. Let $A \subseteq U$ be a nowhere dense set and $x \in \overline{A}$. If $x \in \overline{A \setminus \{x\}}$ then $x \in Z$ and thus $x \notin U$. Therefore A is closed discrete in U. On the other hand, if A is a discrete subset of U then A is obviously nowhere dense. $\square$

Theorem 8.5. *Let τ be a Hausdorff topology over a countable set X without isolated points and $\mathcal{I}$ be an analytic p-ideal over X. Then $NWD(\tau) \not\subseteq \mathcal{I}$. If moreover τ is analytic, then $NWD(\tau)$ is not a p-ideal.*

Proof: We will use the representation of analytic p-ideals in terms of submeasures given by Solecki [20]. A map $\varphi : \mathcal{P}(\mathbf{N}) \to [0, +\infty]$ is a submeasure if $\varphi(\emptyset) = 0$ and $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for all $A, B \subseteq \mathbf{N}$. It is lower semicontinuous if $\varphi(A) = \lim_n \varphi(A \cap \{0, \cdots, n\})$ for all $A \subseteq \mathbf{N}$. Finally, define $\varphi^*(A) = \lim_n \varphi(A \setminus \{0, \cdots, n\})$. The map φ^* satisfies that $\varphi^*(A \triangle B) \leq \varphi^*(A) + \varphi^*(B)$, in particular φ^* is subadditive. The result of Solecki [20] is that every analytic p-ideal has the form

$$\mathrm{Exh}(\varphi) = \{A \subseteq \mathbf{N} : \ \varphi^*(A) = 0\} \tag{4}$$

for some lower semicontinuous submeasure φ on $\mathbf{N}$. To show that $NWD(\tau) \not\subseteq \mathcal{I}$ it suffices to construct a nowhere dense set F such that $\varphi^*(F) > 0$ where φ is a lower semicontinuous submeasure on X representing $\mathcal{I}$ as $\mathrm{Exh}(\varphi)$. To that end, we will consider two cases:

Case 1: Suppose there is $x \in X$ and $\epsilon > 0$ such that $\varphi^*(U) \geq \epsilon$ for all open set U with $x \in U$. Let $\{x_n\}$ be an enumeration of $X \setminus \{x\}$. Fix an open set V_n such that $x_n \in V_n$ and $x \notin \overline{V_n}$. Let $U_n = X \setminus \bigcup_{l \leq n} \overline{V_l}$. The set U_n is an open neighbourhood of x, thus $\varphi^*(U_n) \geq \epsilon$. By the lower semicontinuity of φ and (4) we can find finite sets $F_n \subseteq U_n$ such that $\varphi(F_n) \geq \epsilon/2$ and $F_n \cap F_m = \emptyset$ for $n \neq m$. Let $F = \bigcup_n F_n$. Note that $\varphi^*(F) \geq \epsilon/2$ (this follows from the monotonicity of φ and the fact that φ^* is invariant under finite changes). Since $F \cap V_n \cap V$ is finite for any open set V, then F is nowhere dense.

Case 2: Suppose for all $x \in X$ and all $\epsilon > 0$ there is an open set U such that $\varphi^*(U) < \epsilon$ and $x \in U$. Let $\delta = \varphi^*(X) > 0$ and fix an enumeration $\{x_n\}_{n=0}^{\infty}$ of X and an open neighborhood U_n of x_n such that $\varphi^*(U_n) < \delta \cdot 2^{n+2}$. Let $D_n = X \setminus \bigcup_{l<n} U_l$. Using the subaditivity of φ^* we have that $\varphi^*(D_n) > \frac{\delta}{2}$. Therefore there are finite sets $F_n \subseteq D_n$ such that $\varphi(F_n) > \frac{\delta}{2}$ and $F_n \cap F_m = \emptyset$ for $n \neq m$. Let $F = \bigcup_n F_n$. Then F is nowhere dense and $\varphi^*(F) > \frac{\delta}{2}$.

For the second claim let us assume, towards a contradiction, that $NWD(\tau)$ is a p-ideal. Then by 8.4 there is a dense open set U such that (3) holds. Since τ is analytic, it is easy to check using (3) that $NWD(\tau|U)$ is an analytic p-ideal. But we have shown above that this is not possible since $\tau|U$ is a Hausdorff topology without isolated points $\square$

9 Some ideals which are not representable by T_2 topologies

We will present in this section some examples of ideals on a countable set X which are not of the form $NWD(\tau)$ for any Hausdorff topology τ on X. For example, we will show that $\emptyset \times \mathrm{FIN}$ is not representable in this way. In fact, we will show a more general result. For an ideal $\mathcal{I}$ on ω let $\emptyset \times \mathcal{I}$ be the ideal over $\omega \times \omega$ given by

$$\emptyset \times \mathcal{I} = \{A \subseteq \omega \times \omega : \text{for all } n, \ \{i : (n, i) \in A\} \in \mathcal{I}\}$$

Proposition 9.1. *If $\mathcal{I}$ is a proper F_σ ideal over ω containing all singletons then $\emptyset \times \mathcal{I}$ is not of the form $NWD(\tau)$ for any Hausdorff topology τ over $\omega \times \omega$*

Proof: Denote $\emptyset \times \mathcal{I}$ by $\mathcal{J}$. To see this, suppose τ is a Hausdorff topology with $\mathcal{J} \subseteq NWD(\tau)$, since FIN $\subseteq \mathcal{J}$ then τ has no isolated points. Let $C = \{0\} \times \omega$, then $C \notin \mathcal{J}$. If $C \in NWD(\tau)$ then we are done. Suppose $C \notin NWD(\tau)$ and let $D \subseteq C$ be such that $V \cap D \notin NWD(\tau)$ for all open V with $V \cap D \neq \emptyset$. It is easy to check that $NWD(\tau) \cap \mathcal{P}(D) = NWD(\tau|D)$, where $\tau|D$ is the relative topology on D. By 8.2 we know that $NWD(\tau) \cap \mathcal{P}(D)$ is not F_σ, but $\mathcal{J} \cap \mathcal{P}(D)$ is F_σ, since it is clearly a copy of $\mathcal{I} \cap \mathcal{P}(D')$ where $D' = \{i : (0, i) \in D\}$. $\square$

Our second example is the ideal $\mathcal{I}_{\omega^2}$ on ω^2 consisting of all subsets of the ordinal ω^2 of order type $< \omega^2$.

Proposition 9.2. *The ideal $\mathcal{I}_{\omega^2}$ is not representable as $NWD(\tau)$ by any Hausdorff topology τ on ω^2.*

Proof: In fact, let τ be a Hausdorff topology such that $\mathcal{I} \subseteq NWD(\tau)$. In particular, every nonempty τ-open set has order type ω^2. Let U_n be a pairwise disjoint sequence of nonempty open sets. For each n choose a subset $A_n \subseteq U_n$ of order type $\omega \cdot n$. Since each A_n is nowhere dense, then $\bigcup_n A_n$ is clearly also nowhere dense but it is not in $\mathcal{I}$. $\square$

Our last example is the ideal of order-scattered subsets of $\mathbf{Q}$, that is to say, the collection of subsets of $\mathbf{Q}$ which contain no order-isomorphic copy of $\mathbf{Q}$.

Proposition 9.3. *The ideal of order-scattered subsets of $\mathbf{Q}$ is not representable as $NWD(\tau)$ for any Hausdorff topology τ over $\mathbf{Q}$.*

Proof: Suppose toward a contradiction that there is a Hausdorff topology τ on $\mathbf{Q}$ such that

$$NWD(\tau) = \{A \subseteq \mathbf{Q} : \ \mathrm{otp}(\mathbf{Q}) \not\leq \mathrm{otp}(A)\}.$$

We will construct two Cantor schemes

$$\{U_s : \ s \in 2^{<\omega}\}$$

and

$$\{I_s : \ s \in 2^{<\omega}\}$$

such that for all $s \in 2^{<\omega}$:

(i) U_s is a τ-open set and I_s is an open interval in $\mathbf{Q}$,

(ii) $I_{s^\frown 0} < I_{s^\frown 1}$,

(iii) $U_s \cap I_s$ is a nonempty non order-scattered set.

Assuming this has been accomplished we will finish the proof. Let $A \subseteq 2^{<\omega}$ be an antichain such that

$$T_A = \{s \in 2^{<\omega} : \ t \not\prec s \text{ for all } t \in A\}$$

is a perfect subtree of $2^{<\omega}$ (take for instance a perfect binary tree T such that its set of branches $[T]$ is nowhere dense in $2^{\mathbf{N}}$ and let A be the minimal elements of $2^{<\omega} \setminus T$). For each $s \in A$ pick $x_s \in U_s \cap I_s$ and form the set

$$N = \{x_s : \; s \in A\}$$

Then N is τ-discrete and therefore τ-nowhere dense. On the other hand, by the choice of intervals I_s and the perfectness of the subtree T_A we infer that N contains an order-isomorphic copy of $\mathbf{Q}$, a contradiction.

The construction is by induction on the length of $s \in 2^{<\omega}$. Let $I_\emptyset = U_\emptyset = \mathbf{Q}$. Suppose U_s and I_s has been chosen. By inductive assumption $X_s = I_s \cap U_s$ is not order-scattered so there must be $q \in X_s$ such that $X_s \cap (-\infty, q)$ and $X_s \cap (q, +_\infty)$ are both non order-scattered. Let $I_{s^\frown 0} = I_s \cap (-\infty, q)$ and $I_{s^\frown 1} = I_s \cap (q, +\infty)$. Then $X_s \cap I_{s^\frown 0}$ and $X_s \cap I_{s^\frown 1}$ are two non order-scattered subsets of $\mathbf{Q}$ and therefore two non τ-nowhere-dense sets. Since τ is Hausdorff we can find two disjoint τ-open sets $U_{s^\frown 0}$ and $U_{s^\frown 1}$ such that $I_{s^\frown 0} \cap U_{s^\frown 0}$ and $I_{s^\frown 1} \cap U_{s^\frown 1}$ are both non τ-nowhere dense and therefore both non order-scattered. This finishes the inductive step and the proof of the proposition. □

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