Cardinality and Compactness, Cofinality and Equicontinuity.

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Abstract

A collection of topologies Φ_α (for α an ordinal) is introduced in the space of bounded continuous functions $C_b(X)$ (where X is a discrete space). It is proved that $|X| \leq \aleph_\alpha$ if and only if the unit ball $B_1(X)$ in $C_b(X)$ is Φ_α-compact. We compute the dual of $(C_b(X), \Phi_\alpha)$ and present a characterization of the cofinality of $|X|$ in terms of Φ_0-equicontinuity.

1 Introduction

The compactness (with respect to various topologies) of the unit ball in the space of bounded continuous real functions over a completely regular Hausdorff space X has been used very successfully to characterize some topological properties of X (see [9], [8] [6] and [7]). We introduce a collection of topologies in $C_b(X)$ Φ_α (for α an ordinal) and use them to characterize the cardinality of X and the cofinality of $|X|$. The κ-product topology on 2^κ (which coincides with Φ_0 in 2^κ as a subspace of $C_b(\kappa)$) has been used to study some large cardinal properties of κ (see [2]). In [5] it is shown that the compactness of the unit ball with respect to the strict topologies β_σ and β_p characterize the real measurability and Ulam measurability of $|X|$, respectively. The idea behind the results of this paper is that some set theoretic notions, like cardinality and cofinality, can be characterized using concepts from functional analysis.

The main results are the following:

Theorem A: *Let X be a discrete space. $|X| \leq \aleph_\alpha$ if and only if the unit ball in $C_b(X)$ is Φ_α-compact.*

The proof is a transfinite version of the well known fact that (for X discrete) the unit ball in $C_b(X)$ is $\|\cdot\|$-compact if and only if X is finite.

*Supported by a CDCHT-ULA (Venezuela) grant # C-502-91.

In order to characterize the cofinality of $|X|$ we will look at the dual of $(C_b(X), \Phi_\alpha)$ and then use the notion of Φ_0-equicontinuity. Let E be a locally convex space and κ a cardinal. We say that E is *κ-barrelled* if for every pointwise bounded family $\{\Lambda_i\}_{i\in I}$ in the topological dual of E, with $|I| < \kappa$ we have that $\{\Lambda_i\}_{i\in I}$ is equicontinuous. Put

$$bar(E) = sup\{\kappa : E \text{ is } \kappa\text{-barrelled}\}$$

Theorem B: *Let X be an infinite discrete space. Then $cof(|X|) = bar((C_b(X), \Phi_0))$.*

Now we fix the notation. Let X be a completely regular Hausdorff space, $C_b(X)$ will denote the space of bounded continuous real functions over X and the unit ball $\{f \in C_b(X) : \| f \| \leq 1\}$ will be denoted by $B_1(X)$. If τ and τ' are topologies on some space, $\tau \leq \tau'$ means that τ' is finer than τ. The pointwise topology will be denoted by t_p. All topologies on $C_b(X)$ that we use in this paper are finer than t_p. If $C_b(X)$ is given the supremum norm $\| \cdot \|$, its dual is given by Alexandroff representation theorem and it consists of all finite, finitely additive Baire measures on X (see e.g. [9]). Our set theoretic notation is standard, as in [3]. $|X|$ will denote the cardinality of X. Lower case Greek letters will denote ordinals. Let κ be an infinite cardinal, κ^+ denotes the cardinal succesor of κ, the cofinality of κ (denoted by $cof(\kappa)$) is the least cardinal λ such that there exists a family $\{A_\xi\}_{\xi<\lambda}$ of subsets of κ such that $|A_\xi| < \kappa$ and $\kappa = \bigcup A_\xi$. An infinite cardinal κ is called regular if $cof(\kappa) = \kappa$.

2 Cardinality and compactness

Let X be discrete space, we introduce the cardinal topologies on $C_b(X)$.

Definition 2.1 *Let α be an ordinal and X a set.*

$$\mathcal{S}_\alpha(X) = \{Y \subset X : \textit{either } Y \textit{ is finite or if } |Y| = \aleph_\lambda, \textit{ then } \aleph_{\lambda+\alpha} < |X| \ \}$$

Definition 2.2 *For every $Y \subseteq X$ let $p_Y(f) = \sup\{|f(x)| : \ x \in Y\}$ for $f \in C_b(X)$. p_Y is a seminorm over $C_b(X)$. For every ordinal α, let Φ_α^X be the locally convex topology defined by the family of seminorms $\{p_Y : Y \in \mathcal{S}_\alpha(X)\}$.*

If there is no confusion about X, we will write Φ_α and $\mathcal{S}_\alpha$ instead of Φ_α^X and $\mathcal{S}_\alpha(X)$. We show next some basic facts about the *cardinal topologies* Φ_α. (In [5], another collection of topologies was introduced (also called cardinal topologies) which are finer than the Φ_α's and also satisfy the conclusion of theorem 2.5 below).

Lemma 2.3 *Let X be a discrete space*

1. $t_p \leq \Phi_\alpha \leq \ \| \cdot \|$

2. *If* $\alpha < \beta$ *then* $\Phi_\beta \leq \Phi_\alpha$

3. Φ_α *is the projective topology induced by the maps:*

$$\pi_Y : C_b(X) \to (C_b(Y), \|\cdot\|)$$

where $Y \in \mathcal{S}_\alpha$ *and*

$$\pi_Y(f) = f \mid Y,$$

i.e. Φ_α *is the smallest topology for which the maps* π_Y*'s are continuous. Hence, a net* f_η *in* $C_b(X)$ *converges to zero with respect to* Φ_α *if and only if* f_η *converges uniformly to zero over every set in* $\mathcal{S}_\alpha$

Proof: It follows easily from the definitions. □

The following is a well known fact.

Lemma 2.4 *Let X be a discrete space. X is finite if and only if* $B_1(X)$ *is* $\|\cdot\|$*-compact.* □

Theorem 2.5 *Let X be discrete space. The following statements are equivalent:*

1. $|X| \leq \aleph_\alpha$

2. $\Phi_\alpha = t_p$

3. $B_1(X)$ *is* Φ_α*-compact*

Proof: $(1 \Rightarrow 2)$ If $|X| \leq \aleph_\alpha$ then $\mathcal{S}_\alpha = \{Y \subset X : Y \text{ is finite }\}$. Hence Φ_α is the pointwise topology.

$(2 \Rightarrow 3)$ If $\Phi_\alpha = t_p$ then $B_1(X)$ is Φ_α- compact by the Tychonoff theorem (since X is discrete we have that $B_1(X) = [-1, 1]^X$).

$(3 \Rightarrow 1)$ If $\aleph_\alpha < |X|$, then X contains an infinite countable subset Y and $Y \in \mathcal{S}_\alpha$. Since $\pi_Y : (C_b(X), \Phi_\alpha) \to (C_b(Y), \|\cdot\|)$ is continuous and $\pi_Y(B_1(X)) = B_1(Y)$ we have that $B_1(Y)$ is $\|\cdot\|_Y$-compact and from lemma 2.4 we get that Y is finite, which is a contradiction. □

It is natural to ask if the restriction to discrete spaces is necessary. On this respect we notice the following: Let X be a completely regular Hausdorff space. β_0 is the finest locally convex topology on $C_b(X)$ which coincides on the norm-bounded sets with the compact open topology. Wheeler [8] has proved that X is discrete if and only if $B_1(X)$ is β_0-compact. In fact, a bit stronger result can be shown analogously: if τ is a topology on $C_b(X)$ with $t_p \leq \tau$ and $B_1(X)$ is τ-compact then X is discrete. Thus it is interesting to determine the relationship between β_0 and the cardinal topologies we have defined. This was done in [5] and for the sake of completeness we repeat it here.

Lemma 2.6 *Let X be a discrete space and (f_α) be a net on $C_b(X)$. Then $f_\alpha \to 0$ in β_0 if and only if for every sequence $(x_n) \in X$ and every sequence $(r_n) \in \mathbf{R}$ which converges to zero we have that $Sup\{|r_n f_\alpha(x_n)| : \ n \geq 0\} \to 0$*

Proof: β_0 is also characterized as the topology determinded by the seminorms $p_h(f) = \| hf \|$ where h is a bounded real function on X vanishing at infinity (i.e. $\{x \in X : |h(x)| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$) (see e.g. theorem 2.4 in [4]). Using this characterization for X discrete, it is easy to obtain the result. $\Box$

As a corollary we get

Lemma 2.7 *Let X be a discrete space and τ a topology on $C_b(X)$ such that for every net $(f_\alpha) \in C_b(X)$ with $(f_\alpha) \xrightarrow{\tau} 0$, we have that for every countable $Z \subseteq X$ $\ f_\alpha \mid Z \xrightarrow{\|\cdot\|} 0$. Then $\beta_0 \leq \tau$* $\Box$

We have the following comparison theorem

Theorem 2.8 *Let X be a discrete space. For every ordinal α one and only one of the following holds:*

(a) $t_p = \Phi_\alpha$

(b) $\beta_0 < \Phi_\alpha$

In particular β_0 and Φ_α are always comparable.

Proof: If $|X| \leq \aleph_\alpha$ then by theorem 2.5 we have $\Phi_\alpha = t_p$ and (a) holds. On the other hand, suppose $|X| = \aleph_\beta > \aleph_\alpha$. By the definition of Φ_α the hypothesis of the previous lemma are satisfied, hence $\beta_0 \leq \Phi_\alpha$. By the result of Wheeler mentioned before we get $\beta_0 < \Phi_\alpha$, otherwise $B_1(X)$ would be Φ_α-compact and therefore by theorem 2.5 $|X| \leq \aleph_\alpha$, which is a contradiction. $\Box$

We will see next that the relation $|X| = |Y|$ can be topologically characterized.

Theorem 2.9 *Let X and Y be infinite discrete spaces. $|X| \leq |Y|$ if and only if for every α there is a continuous onto map $T : (C_b(Y), \Phi^Y_\alpha) \to (C_b(X), \Phi^X_\alpha)$.* [1]

Proof: ($\Rightarrow$) Let $h : X \to Y$ be a 1-1 map. Let $T : C_b(X) \to C_b(Y)$ be defined by $T(f) = f \circ h$. Since h is injective it is clear that T is onto. Let $Z \in \mathcal{S}_\alpha(X)$ and $W = h[Z]$ then $p_Z(T(f)) = p_W(f)$, from which follows that T is continuous.

($\Leftarrow$) Let α be such that $|Y| \leq \aleph_\alpha$. We will see that $|X| \leq \aleph_\alpha$. Hence $|X| \leq |Y|$.

[1]As the referee pointed out, this result is false for X and Y finite.

Let T as in the hypothesis. By 2.5 $B_1(Y)$ is Φ_α^Y-compact, hence $C_b(Y)$ is the union of a countable collection of Φ_α^Y-compact sets, then (as T is onto and continuous) $C_b(X)$ is the union of a countable collection of Φ_α^X-compact sets. Since $\Phi_\alpha^X \leq \|\cdot\|$, then those compact sets are norm closed. Hence by the Baire category theorem one of them has non empty interior (in the norm topology) and therefore there is a Φ_α^X-compact ball, which implies that $B_1(X)$ is also Φ_α^X-compact. Hence by 2.5 $|X| \leq \aleph_\alpha$. □

Let us observe that if $h : X \to Y$ is a bijection then the map T defined in the previous proof is a homeomorphism. Hence we have

Corollary 2.10 *Let X and Y be discrete spaces. $|X| = |Y|$ if and only if for every α $(C_b(Y), \Phi_\alpha^Y)$ is homeomorphic to $(C_b(X), \Phi_\alpha^X)$.* [2] □

3 Cofinality and equicontinuity

As mentioned in the introduction, in order to characterize the cofinality of $|X|$ we will look at the dual space of $(C_b(X), \Phi_0)$. Following the classical Alexandroff's theorem, dual spaces are identified with a collection of Baire measures over X.

Theorem 3.1 *(**Alexandroff's Theorem**)(See e.g. [9]) Let X be a completely regular Hausdorff space. Then the map $T : M(X) \to C_b(X)'$ defined by $T(\mu)(f) = \int_X f\, d\mu$ in an isometric isomorphism from $M(X)$ (the finite, finitely additive Baire measures with the total variation norm) onto $C_b(X)$ (with the norm topology).*

We identify first the Φ_α-equicontinuous sets.

Lemma 3.2 *Let X be discrete. A subset $\mathcal{A}$ of $C_b(X)^\star$ (the algebraic dual of $C_b(X)$) is Φ_α-equicontinuous if and only if there exists $r > 0$ and $Y \in \mathcal{S}_\alpha$ such that:*

$$|\Lambda(f)| \leq r p_Y(f) \ \text{ for all } \Lambda \in \mathcal{A} \text{ and all } f \in C_b(X).$$

Proof: The condition is clearly sufficient. For the other direction suppose that $\mathcal{A}$ is a Φ_α-equicontinuous subset of linear functionals on $C_b(X)$. There exists $Y \in \mathcal{S}_\alpha$ and $\delta > 0$ such that for all $\Lambda \in \mathcal{A}$, if $p_Y(f) \leq \delta$ then $|\Lambda(f)| \leq 1$. It is easy to see that $r = 1/\delta$ and Y work. (Observe that if $f \in C_b(X)$, $\Lambda \in \mathcal{A}$ and $p_Y(f) = 0$ then $\Lambda(f) = 0$. Otherwise $|\Lambda(\delta f/p_Y(f))| \leq 1$). □

Now we will identify the measures representing the dual of $(C_b(X), \Phi_\alpha)$.

Lemma 3.3 *$\Lambda \in (C_b(X), \Phi_\alpha)'$ if and only if there exists a unique $\mu \in M(X)$ such that for some $Y \in \mathcal{S}_\alpha$ with $|\mu|(Y) = |\mu|(X)$*

$$\Lambda(f) = \int f d\mu \quad \text{for all} \quad f \in C_b(X).$$

[2] ver comentario del referee

Proof: ($\Rightarrow$) Let $\mu \in M(X)$ be the measure given by the Alexandroff's theorem and let $Y \in \mathcal{S}_\alpha$ and $r > 0$ given by 3.2 be such that

$$|\Lambda(f)| \leq r p_Y(f) \text{ for all } f \in C_b(X).$$

We will show that $|\mu|(X \setminus Y) = 0$. Let $\chi_B \in C_b(X)$ be the characteristic function of $B \subset X \setminus Y$ then

$$|\int \chi_B d\mu| = |\Lambda(\chi_B)| = 0$$

for all $B \subset X \setminus Y$. Hence $|\mu|(X \setminus Y) = 0$.

($\Leftarrow$) Let $Y \in \mathcal{S}_\alpha$ such that $|\mu|(X) = |\mu|(Y)$, then

$$|\Lambda(f)| = \left|\int f d\mu\right| \leq \left|\int_Y f d\mu\right| + \left|\int_{X\setminus Y} f d\mu\right| = \left|\int_Y f d\mu\right|.$$

Thus $|\Lambda(f)| \leq |\mu|(Y) p_Y(f)$ for all $f \in C_b(X)$. □

It is a well known fact that a locally convex space X is *barrelled* if and only if every pointwise bounded family of continuous linear functionals over X is equicontinuous (see e.g. [1]). We will introduce a "cardinal version" of this property which will be used to characterize cofinality.

Definition 3.4 *Let E be locally convex space and κ cardinal. We say that E is κ-*barrelled *if for every pointwise bounded family $\{\Lambda_i\}_{i\in I}$ in the topological dual of E such that $|I| < \kappa$ we have that $\{\Lambda_i\}_{i\in I}$ is equicontinuous.*

Definition 3.5 *Let E be a locally convex space. Then*

$$bar(E) = \sup\{\kappa : E \text{ is } \kappa\text{-barrelled}\}$$

Theorem 3.6 *Let X be an infinite discrete space. Then*

$$bar(C_b(X), \Phi_0) = cof(|X|).$$

Proof: Let $|X| = \kappa$ and $cof(\kappa) = \lambda$. We will show first that if $\{\Lambda_i\}_{i\in I}$ is a pointwise bounded family of linear Φ_0-continuous functionals in $C_b(X)$ and $|I| < \lambda$ then $\{\Lambda_i\}_{i\in I}$ is Φ_0-equicontinuous, i.e. $(C_b(X), \Phi_0)$ is λ-barrelled.

From 3.2 we know that for every $i \in I$ there exists $Y_i \subset X$ with $|Y_i| < \kappa$ and $r_i > 0$ such that

$$|\Lambda_i(f)| \leq r_i p_{Y_i}(f) \text{ for all } f \in C_b(X).$$

Let $Y = \bigcup_{i\in I} Y_i$, then $Y \in \mathcal{S}_0$ and

$$\Lambda_i(f)| \leq r_i p_Y(f) \text{ for all } f \in C_b(X). \tag{1}$$

We define $\widehat{\Lambda_i} : C_b(Y) \to \mathbf{R}$ by $\widehat{\Lambda_i}(f) = \Lambda_i(\bar{f})$ where

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\{\widehat{\Lambda_i} : i \in I\}$ is a subset of $(C_b(Y), \| \ \|)'$ and $\{\widehat{\Lambda_i} : i \in I\}$ is pointwise bounded. By the Banach-Steinhaus's Theorem there exists $r > 0$ such that $|\widehat{\Lambda_i}(f)| \leq rp_Y(f)$ for all $f \in C_b(Y)$ and all $i \in I$. From (1) we have that if $f \in C_b(X)$ and $f|Y = 0$, then $\Lambda_i(f) = 0$. In particular we have that $\Lambda_i(f) = \widehat{\Lambda_i}(f \mid Y)$ for every $f \in C_b(X)$ and every $i \in I$ and now the result follows.

To finish the proof we will show that $(C_b(X), \Phi_0)$ is not λ^+-barrelled. Let $\{Y_\xi : \xi < \lambda\}$ be a family of pairwise disjoint infinite subsets of X such that $X = \bigcup_{\xi<\lambda} Y_\xi$ and $|Y_\xi| < |X|$. Let $\{\mu_\xi : \xi < \lambda\}$ be finitely additive measures on X with values in $\{0, 1\}$ such that $\mu_\xi(Y_\xi) = 1$ and $\mu_\xi(Y) = 0$ for $Y \subset Y_\xi$ with $|Y| < |Y_\xi|$ (i.e., μ_ξ is a uniform ultrafilter over Y_ξ).

Let $\Lambda_\xi \in (C_b(X), \Phi_0)'$ such that

$$\Lambda_\xi(f) = \int f d\mu_\xi$$

for $f \in C_b(X)$ and $\xi < \lambda$.

It is clear that $\{\Lambda_\xi : \xi < \lambda\}$ is pointwise bounded. We will show that it is not Φ_0-equicontinuous. Suppose, towards a contradiction, that there is $Y \subset X$ with $|Y| < |X|$ and $r > 0$ such that

$$|\Lambda_\xi(f)| \leq rp_Y(f)$$

for all $f \in C_b(X)$ and $\xi < \lambda$.

In particular, if f is the characteristic function of $Y_\xi \setminus Y$ then $\Lambda_\xi(f) = \mu_\xi(Y_\xi \setminus Y)$. Since $p_Y(f) = 0$ then $\mu_\xi(Y_\xi \setminus Y) = 0$ and $\mu_\xi(Y_\xi \cap Y) = \mu_\xi(Y) = 1$ for all $\xi < \lambda$. Then $|Y_\xi \cap Y| = |Y_\xi|$ for all $\xi < \lambda$ and

$$|Y| = \sum_{\xi<\lambda} |Y \cap Y_\xi| = \sum_{\xi<\lambda} |Y_\xi| = |X|$$

which is a contradiction. □

As a corollary we immediately get

Corollary 3.7 *Let X be an infinity discrete space and $|X| = \kappa$, then κ is regular if and only if $(C_b(X), \Phi_0)$ is κ-barrelled.* □

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Acknowledgment: We are very grateful to the referee for all his valuable sugestions that improved the presentation of the paper.

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