Low separation axioms via the diagonal

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Abstract

In the context of a generalized topology $\mathbf{g}$ on a set X, we give in this article characterizations of some separation axioms between T_0 and T_2 in terms of properties of the diagonal in $X \times X$.

Key Words: Generalized topologies, intersection structures, envelopes, kerneled and saturated sets.
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1 Introduction

A well known elementary fact says that a topological space X is Hausdorff iff the diagonal Δ is closed in $X \times X$. In this paper we show that behind this observation there is a general pattern which includes several separation axioms below T_2 (namely T_0, $T_{1/4}$, $T_{1/2}$, T_1, R_0 and R_1). These low separation axioms have been studied in a more general setting where, instead of open sets, other kind of subsets are used: semi-open sets, α-open sets, λ-open sets, etc. ([1], [6], [8]). These families (called generalized topologies in [5]) always contain $\emptyset$ and X and are closed under arbitrary unions (but not necessarily under finite intersections). On the other hand, in the study of low separation axioms, set operations similar to the closure operator are frequently used. These operations are naturally extended to the context of a generalized topology $\mathbf{g}$ (and are then called envelopes [5]). For instance, $k_g(A)$ corresponds to the topological closure of A, $\chi_g(A)$ corresponds to the kernel of A [7] (*i.e.* the intersection of all open sets containing A) and $sat_g(A)$ correspond to the union of the closure of points in A. Our characterizations

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of the separation axioms are in terms of the behavior of Δ under k_g, χ_g and sat_g. An example of our results is that $\mathbf{g}$ satisfies T_1 iff $\chi_g(\Delta) = \Delta$.

We will also give a characterization of low separation axioms in terms of saturated sets. A set in a topological space is said to be saturated when it contains the closure of each of its points. It is known that a topology satisfies the axiom R_0 iff every open set is saturated [5]. Another notion of saturation was studied in [4]. We extend these notions and study its connection with low separation axioms.

The paper is organized as follows. In section 2 we recall the basic separation axioms and state some facts about generalized topologies and envelope operations. The results about the properties of the diagonal and the separation axioms are shown in section 3. In section 4 we study the family of saturated sets and its connection with the separation axioms. Finally, in section 5 we analyze the axioms $T_{1/2}$ and $T_{1/4}$.

2 Preliminaries

We follow the notations and definitions used in [5]. A subset $\mathbf{g}$ of the power set $\mathcal{P}(X)$ of a set X is a *generalized topology* (briefly GT) on X if $\{\emptyset,\ X\} \subseteq \mathbf{g}$ and $\mathbf{g}$ is closed under arbitrary unions. If $\mathbf{g}$ is a generalized topology, then the family of complements of sets in $\mathbf{g}$ is usually called an intersection structure. In this article $\mathbf{g}$ will always denote a generalized topology.

Definition 2.1 *[5] An envelope operation on X is a mapping $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ such that*

(i) $A \subseteq \rho A$ *for* $A \subseteq X$.

(ii) If $A \subseteq B$, *then* $\rho A \subseteq \rho B$ *for all* $A \subseteq B \subseteq X$.

(iii) $\rho A = \rho\rho A$ *for* $A \subseteq X$.

More generally, $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ *is called a weak envelope if* (i) *and* (ii) *are satisfied.*

Examples of envelope operations are given below.

Definition 2.2 *Let* $A \subseteq X$.

(i) $\chi_g(A) = \bigcap \{H \in \mathbf{g} :\ A \subseteq H\}$.

(ii) $k_g(A) = \{x \in X :\ K \cap A \neq \emptyset$ *for each* $K \in \mathbf{g}$ *with* $x \in K\}$.

(*iii*) $sat_g(A) = \bigcup_{x \in A} k_g(\{x\})$.

The operator χ_g and k_g were defined in [5] and shown to be envelope operations. It is straightforward to show that sat_g is an envelope. It is also easy to see that $\chi_g(A) = A$ for all $A \in \mathbf{g}$. Moreover, $x \in \chi_g(y)$ if and only if $y \in k_g(x)$ for any $x, y \in X$ (where we write $\rho(x)$ instead of $\rho(\{x\})$ for any set operator ρ). When $\mathbf{g}$ is a topology, k_g is the closure operator cl and $\chi_g(A)$ is the kernel of A, frequently denoted by $\widehat{A}$ or $Ker(A)$. Notice that, in general, if τ is the topology generated by a GT $\mathbf{g}$, then $k_g \neq cl_\tau$ and $\chi_g \neq Ker_\tau$ (for instance, in $\mathbb{R}$ take $\mathbf{g}$ to be the GT generated by the collection of intervals of the form $(-\infty, a)$ and $(a, +\infty)$).

Now we formulate the fundamental separation axioms in terms of an arbitrary GT ([5]).

($\mathbf{T}_0$) For all $x, y \in X$, with $x \neq y$ there is $K \in \mathbf{g}$ containing precisely one of x and y.

($\mathbf{T}_1$) For all $x, y \in X$ with $x \neq y$ there is $K \in \mathbf{g}$ such that $x \in K$, $y \notin K$.

($\mathbf{T}_2$) For all $x, y \in X$, $x \neq y$, there are $K, K' \in \mathbf{g}$ such that $x \in K$, $y \in K'$ and $K \cap K' = \emptyset$.

($\mathbf{R}_0$) For all $x, y \in X$, if $k_g(x) \neq k_g(y)$, then $k(x) \cap k_g(y) = \emptyset$.

($\mathbf{R}_1$) For all $x, y \in X$, if $k_g(x) \neq k_g(y)$, then there are $K, K' \in \mathbf{g}$ disjoint such that $k_g(x) \subseteq K$ and $k_g(y) \subseteq K'$.

Proposition 2.3 *[5]* $\mathbf{g}$ *satisfies* ($\mathbf{R}_0$) *iff for all* $x, y \in X$*, if there is* $K \in \mathbf{g}$ *such that* $x \in K$ *and* $y \notin K$*, then there is* $K' \in \mathbf{g}$ *such that* $x \notin K'$ *and* $y \in K'$*.*

Two of the recently widely studied separation axioms below T_1 can be stated for a generalized topology $\mathbf{g}$ as follows:

($\mathbf{T_{1/2}}$) For all $x \in X$, $\{x\} \in \mathbf{g}$ or $\{x\} = k_g(x)$.

($\mathbf{T_{1/4}}$) For all $x \in X$, $\{x\} = \chi_g(x)$ or $\{x\} = k_g(x)$.

In the rest of this section we introduce some notions and present some basic facts about the envelopes χ_g, k_g and sat_g that will be used in the sequel. In order to simplify the notation, we will write $\chi(A)$, $k(A)$ and $sat(A)$ avoiding the use of $\mathbf{g}$.

We say that a set A is *closed* (resp. *kerneled*) iff $k(A) = A$ (resp. $\chi(A) = A$). When $\mathbf{g}$ is a topology, kerneled sets are usually called Λ-sets [7]. A subset $A \subseteq X$ is said $\mathbf{g}$-*saturated* (or just saturated) if $sat(A) = A$, equivalently if $k(x) \subseteq A$ for all $x \in A$. Note that $sat(A)$ is the smallest saturated set containing A, and that A is saturated if and only if $X \backslash A$ is kerneled, where $X \backslash A$ denotes the complement of A. In particular $sat(x) = k(x)$, for any $x \in X$. The collection of all saturated subsets of X is denoted by $S(X)$. It is easy to see that $S(X)$ is closed under arbitrary unions and arbitrary intersections.

Proposition 2.4 *[5] A is closed iff $X \backslash A \in \mathbf{g}$.*

Proof. Let A closed. If $y \in X \backslash A$ then $y \notin k(A)$, and hence there exists $B_y \in \mathbf{g}$ such that $y \in B_y$ and $B_y \cap A = \emptyset$. Thus $X \backslash A = \bigcup_{y \in X \backslash A} B_y \in \mathbf{g}$. The converse is obvious. ■

Since our analysis of the separation axioms will be in terms of the behavior of the diagonal Δ, we need to introduce the GT on the product $X \times X$. Let $\mathbf{g}$ be a GT on X, then the family $\mathbf{g}^2$ below is a GT in X^2:

$$\mathbf{g}^2 = \left\{ D \subseteq X \times X : \ D = \bigcup_{\alpha} A_\alpha \times B_\alpha, \text{ with } A_\alpha, \ B_\alpha \in \mathbf{g} \right\}.$$

In this article the operators k, χ and sat on $X \times X$ refer to the generalized topology $\mathbf{g}^2$.

Proposition 2.5 *For any $(x, y) \in X \times X$ the following holds:*

(i) $\chi(x, y) = \chi(x) \times \chi(y)$.

(ii) $k(x, y) = k(x) \times k(y)$.

Proof. (i) Let $(p, q) \in \chi(x, y)$ and $A, B \in \mathbf{g}$ with $x \in A$ and $y \in B$. Then $(x, y) \in D = A \times B \in \mathbf{g}^2$ and so $(p, q) \in A \times B$. Thus $p \in \chi(x)$ and $q \in \chi(y)$. Conversely, if $(p, q) \in \chi(x) \times \chi(y)$ and $D \in \mathbf{g}^2$, then $(x, y) \in D = \bigcup_\alpha A_\alpha \times B_\alpha$, with A_α, $B_\alpha \in \mathbf{g}$. There is α such that $x \in A_\alpha$ and $y \in B_\alpha$ and hence $(p, q) \in A_\alpha \times B_\alpha \subseteq D$. This implies that $(p, q) \in \chi(x, y)$. ■

Proposition 2.6 *(i) If $A = \bigcup_{i \in I} A_i$, then $\chi(A) = \bigcup_{i \in I} \chi(A_i)$.*

(ii) If $A = \bigcup_{i \in I} A_i$, then $sat(A) = \bigcup_{i \in I} sat(A_i)$.

Proof. (i) was proved in [5] and (ii) is obvious. ■

Proposition 2.7 *Let A be a subset of $X \times X$. Then*
(i) $\chi(A) = \bigcup_{(x,y)\in A} \chi(x) \times \chi(y)$.
(ii) $sat(A) = \bigcup_{(x,y)\in A} k(x) \times k(y)$.

Proof. The result follows directly from propositions 2.5 and 2.6. ■

To end this section, we introduce three more operations. Let $A \subseteq X$, then define

$$\begin{aligned} k_\theta(A) &= \{x \in X :\ A \cap k(D) \neq \emptyset \text{ for each } D \in \mathbf{g} \text{ such that } x \in D\} \\ k_\lambda(A) &= k(A) \cap \chi(A) \\ k_\mu(A) &= sat(A) \cap \chi(A) \end{aligned}$$

It is easy to see that k_θ is a weak envelope on X such that $k(A) \subseteq k_\theta(A)$ for all $A \subseteq X$. Also, it is straightforward to show that k_λ and k_μ are envelopes on X (more generally, the finite intersection of envelopes is again an envelope). When $\mathbf{g}$ is a topology, k_θ is the well known cl_θ operator [9, 4] and k_λ the cl_λ operator [2]. The cl_λ-closed sets (*i.e.* sets such that $cl_\lambda(A) = A$) are usually called λ-closed sets and their complements λ-open sets [1]. If $\mathbf{g}$ is the GT consisting of the λ-open sets, then $k = cl_\lambda$.

3 Separation axioms as properties of the diagonal

We will denote by Δ the diagonal in $X \times X$. In this section we show that the separation axioms can be characterized in terms of $\chi(\Delta)$, $sat(\Delta)$ and $k(\Delta)$. Besides Δ there are two others binary relations which play an important role in what follows.

$$\begin{aligned} (x,y) \in L_g \quad &\textit{iff} \quad \forall A \in \mathbf{g} \quad [x \in A \rightarrow y \in A] \\ (x,y) \in E_g \quad &\textit{iff} \quad \forall A \in \mathbf{g} \quad [x \in A \leftrightarrow y \in A]\,. \end{aligned}$$

Notice that $\Delta \subseteq E_g \subseteq L_g$. Moreover, L_g is transitive relation and E_g is an equivalence relation on X.

The main result of this section is summarized in the following table.

T_2	$\Leftrightarrow$	$k(\Delta)$	$=$	Δ				
T_1	$\Leftrightarrow$	$\chi(\Delta)$	$=$	Δ	$\Leftrightarrow$	$sat(E_g)$	$=$	Δ
T_0	$\Leftrightarrow$	Δ	$=$	E_g				
R_0	$\Leftrightarrow$	$\chi(\Delta)$	$=$	E_g	$\Leftrightarrow$	E_g	$=$	$sat(\Delta)$
R_1	$\Leftrightarrow$	$k(\Delta)$	$=$	E_g				

In order to show these results we need several auxiliary lemmas.

Lemma 3.1 *(i)* $(x, y) \in L_g$ *iff* $y \in \chi(x)$ *iff* $x \in k(y)$.

(ii) $(x, y) \in E_g$ *iff* $k(x) = k(y)$ *iff* $\chi(x) = \chi(y)$.

Proof. Since $\chi(x) = \cap\{A \in \mathbf{g} : x \in A\}$, (i) follows directly from the definition of L_g. Part (ii) follows from the symmetry of the relation E_g. ■

The following result characterizes χ, k, and sat for the diagonal Δ on $X \times X$. In particular, it shows that $k(\Delta)$, $\chi(\Delta)$ and $sat(\Delta)$ are symmetric (and obviously reflexive) relations.

Lemma 3.2 *(i)* $(x, y) \in k(\Delta)$ *iff* $A \cap B \neq \emptyset$ *for all* $A, B \in \mathbf{g}$ *such that* $x \in A$ *and* $y \in B$.

(ii) $(x, y) \in \chi(\Delta)$ *iff* $k(x) \cap k(y) \neq \emptyset$.

(iii) $(x, y) \in sat(\Delta)$ *iff* $\chi(x) \cap \chi(y) \neq \emptyset$.

Proof. (i) Let $(x, y) \in k(\Delta)$ and let $A, B \in \mathbf{g}$ with $x \in A$ and $y \in B$. Then $(x, y) \in A \times B \in \mathbf{g}^2$ and thus $A \times B \cap \Delta \neq \emptyset$, which implies $A \cap B \neq \emptyset$. Reciprocally, let $D \in \mathbf{g}^2$ containing (x, y). Then $D = \bigcup_\alpha A_\alpha \times B_\alpha$, with A_α, $B_\alpha \in \mathbf{g}$. It follows that $(x, y) \in A_\alpha \times B_\alpha$ for some α. By assumption $A_\alpha \cap B_\alpha \neq \emptyset$. If $z \in A_\alpha \cap B_\alpha$, then $(z, z) \in D \cap \Delta$. Therefore $(x, y) \in k(\Delta)$.

(ii) $(x, y) \in \chi(\Delta) = \bigcup_{x \in X} \chi(x) \times \chi(x)$ if and only if there is $z \in X$ such that $(x, y) \in \chi(z) \times \chi(z)$ for some $z \in X$ if and only if $(z, z) \in k(x) \times k(y)$.

(iii) Follows by a similar argument as (ii). ■

From lemma 3.2(i), it follows that $(x, y) \in k(\Delta)$ iff $\forall A \in \mathbf{g}$ $[x \in A \to y \in k(A)]$ iff $\forall B \in \mathbf{g}$ $[y \in B \to x \in k(B)]$. Therefore we have the following fact about the operator k_θ (defined in section 2).

Lemma 3.3 $(x, y) \in k(\Delta)$ *iff* $y \in k_\theta(x)$ *iff* $x \in k_\theta(y)$.

We prove that the envelope operations k, χ and sat coincide on the relations Δ, L_g and E_g.

Lemma 3.4 *(i)* $k(\Delta) = k(L_g) = k(E_g)$.

(ii) $\chi(\Delta) = \chi(L_g) = \chi(E_g)$.

(iii) $sat(\Delta) = sat(L_g) = sat(E_g)$.

Proof. (i). Since $\Delta \subseteq E_g \subseteq L_g$, it suffices to show that $L_g \subseteq k(\Delta)$. Let $(x,y) \in L_g$ and let $A, B \in \mathbf{g}$ with $(x,y) \in A \times B$. By lemma 3.1, $x \in k(y)$ and thus $y \in \chi(x)$. Since $\chi(x) \subseteq A$ then $y \in A$. It follows that $(y,y) \in A \times B$. Therefore $(x,y) \in k(\Delta)$, by definition of $k(\Delta)$.

(ii). As in (i) we only show that $L_g \subseteq \chi(\Delta)$. Let $(x,y) \in L_g$. By proposition 2.7, $\chi(\Delta) = \bigcup_{z \in X} \chi(z) \times \chi(z)$. Thus, if $(x,y) \notin \chi(\Delta)$, then in particular $y \notin \chi(x)$ and this implies that there is $A \in \mathbf{g}$ containing x such that $y \notin A$, a contradiction.

(iii). If $(x,y) \in L_g$ then, by lemma 3.1(i) and proposition 2.7, $(x,y) \in k(y) \times k(y) \subseteq sat(\Delta)$. Hence $L_g \subseteq sat(\Delta)$. ■

Proposition 3.5 (i) $\mathbf{g}$ *satisfies* $(\mathbf{T}_2)$ *iff* $k_\theta(x) = \{x\}$ *for each* $x \in X$.

(ii) $\mathbf{g}$ *satisfies* $(\mathbf{T}_1)$ *iff* $k(x) = \{x\}$ *for each* $x \in X$ *iff* $\chi(x) = \{x\}$ *for each* $x \in X$.

(iii) $\mathbf{g}$ *satisfies* $(\mathbf{T}_0)$ *iff* $k_\lambda(x) = \{x\}$ *for each* $x \in X$. *That is to say,* $k(x) \cap \chi(x) = \{x\}$ *for each* $x \in X$.

Proof. (i) First note that, if $A, B \in \mathbf{g}$, then $A \cap B = \emptyset$ iff $A \cap k(B) = \emptyset$. Suppose $\mathbf{g}$ satisfies $(\mathbf{T}_2)$. Given $x \in X$ and $y \neq x$, there exist A, $B \in \mathbf{g}$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$, then $y \notin k_\theta(A)$ and, in particular, $y \notin k_\theta(x)$. Therefore $k_\theta(x) = \{x\}$ for each $x \in X$. Conversely, suppose $k_\theta(x) = \{x\}$ for each $x \in X$. Given $x, y \in X$, if $x \neq y$ then $y \notin k_\theta(x)$. Thus $(x,y) \notin k(\Delta)$ and hence, by lemma 3.2, there exist A, $B \in \mathbf{g}$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$, which shows that $\mathbf{g}$ satisfies $(\mathbf{T}_2)$.

(ii) $\mathbf{g}$ satisfies $(\mathbf{T}_1)$ iff given $x \in X$ and $y \neq x$, there exist $A \in \mathbf{g}$ such that $x \in A$ and $y \notin A$, iff given $x \in X$ and $y \neq x$, $y \notin k(x)$, iff $k(x) = \{x\}$ for each $x \in X$. For the second part, note that if $k(x) = \{x\}$ for each $x \in X$, then the set $X \backslash \{x\} = \bigcup_{y \neq x} k(y)$ is saturated for each $x \in X$, and thus $\{x\}$ is kerneled for each $x \in X$. A similar argument shows the reverse implication.

(iii) $\mathbf{g}$ satisfies $(\mathbf{T}_0)$ iff given $x \in X$ and $y \neq x$, $y \notin k(x)$ or $x \notin k(y)$, iff $y \notin k(x)$ or $y \notin \chi(x)$, iff $k_\lambda(x) = k(x) \cap \chi(x) = \{x\}$ for each $x \in X$. ■

Now we start showing the main results of this section.

Theorem 3.6 (i) $\mathbf{g}$ *satisfies* $(\mathbf{T}_2)$ *iff* $k(\Delta) = \Delta$ *iff* Δ *is closed.*

(ii) $\mathbf{g}$ *satisfies* $(\mathbf{T}_1)$ *iff* $\chi(\Delta) = \Delta$ *iff* $L_g = \Delta$ *iff* Δ *is saturated iff* Δ *is kerneled.*

(iii) $\mathbf{g}$ *satisfies* $(\mathbf{T}_0)$ *iff* $E_g = \Delta$.

Proof. (i) By proposition 3.5(i), $\mathbf{g}$ satisfies $(\mathbf{T}_2)$ iff $k_\theta(x) = \{x\}$ for each $x \in X$, iff $y \neq x$ implies $y \notin k_\theta(x)$ iff $(x, y) \notin k(\Delta)$. The second part is obvious.

(ii) Suppose $\mathbf{g}$ satisfies $(\mathbf{T}_1)$. From proposition 3.5(ii), $k(x) = \{x\}$ for each $x \in X$. If $(x, y) \in \chi(\Delta)$, then $k(x) \cap k(y) \neq \emptyset$ and thus $x = y$. On the other hand, if $\chi(\Delta) = \Delta$ and $x \neq y$, then $(x, y) \notin \chi(\Delta)$ and thus $k(x) \cap k(y) = \emptyset$. In particular $x \notin k(y)$, so there exists $A \in \mathbf{g}$ such that $x \in A$ and $y \notin A$. Therefore $\mathbf{g}$ satisfies $(\mathbf{T}_1)$. The second and third parts follow from lemma 3.1(i) and proposition 3.5(ii). The last part is obvious.

(iii) $\mathbf{g}$ satisfies $(\mathbf{T}_0)$ iff for all $x \neq y$, $y \notin \chi(x)$ or $x \notin \chi(y)$ iff $\chi(x) \neq \chi(y)$ iff $(x, y) \notin E_g$ iff $E_g = \Delta$. ■

Theorem 3.7 $\mathbf{g}$ *satisfies* $(\mathbf{R}_0)$ *iff* $\chi(\Delta) = E_g$ *iff* $\chi(\Delta) = L_g$ *iff* E_g *is kerneled iff* E_g *is saturated.*

Proof. Since $E_g \subseteq \chi(E_g) = \chi(\Delta)$, then $\chi(\Delta) = E_g$ iff $\chi(\Delta) \subseteq E_g$. From proposition 2.3, $\mathbf{g}$ satisfies $(\mathbf{R}_0)$ iff $x, y \in X$ implies $k(x) = k(y)$ or $k(x) \cap k(y) = \emptyset$. Therefore $\mathbf{g}$ satisfies $(\mathbf{R}_0)$ iff $\chi(\Delta) = E_g$. The second part of the equivalence follows from the fact that $E_g \subseteq L_g \subseteq \chi(L_g) = \chi(\Delta)$. The third part is obvious. On the other hand, since $y \in k(x)$ iff $x \in \chi(y)$, then $\mathbf{g}$ satisfies $(\mathbf{R}_0)$ iff the sets $\chi(x)$, $x \in X$, form a partition of X, iff $sat(\Delta) = E_g$. ■

Theorem 3.8 $\mathbf{g}$ *satisfies* $(\mathbf{R}_1)$ *iff* $k(\Delta) = E_g$ *iff* $k(\Delta) = L_g$ *iff* E_g *is closed.*

Proof. $\mathbf{g}$ satisfies $(\mathbf{R}_1)$ iff $x, y \in X$, and $k(x) \neq k(y)$, implies the existence of $A, B \in \mathbf{g}$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$ iff $(x, y) \notin E_g$ implies $(x, y) \notin k(\Delta)$ iff $k(\Delta) \subseteq E_g$. Since $E_g \subseteq k(E_g) = k(\Delta)$, it follows that $\mathbf{g}$ satisfies $(\mathbf{R}_1)$ iff $k(\Delta) = E_g$. The other two equivalences are obvious. ■

Corollary 3.9 *(i)* $\mathbf{g}$ *satisfies* $(\mathbf{R}_0)$ *iff* $k(x) = \chi(x)$ *for each* $x \in X$.

(ii) $\mathbf{g}$ *satisfies* $(\mathbf{R}_1)$ *iff* $k_\theta(x) = \chi(x) = k(x)$ *for each* $x \in X$.

Proof. (i) and (ii) follow from theorems 3.7 and 3.8 respectively. ■

Remark 3.10 *If* X *is a topological space, and* $\mathbf{g}$ *is the family of the* λ-*open sets, then* $k_g(x)$ *and* $\chi_g(x)$ *are usually denoted* $cl_\lambda(x)$ *and* $\lambda ker(x)$

respectively [3]. These envelopes satisfy that $cl_\lambda(x) = \lambda Ker(x)$ for all $x \in X$. In fact, since every open set and every closed set is λ-open, then $\lambda Ker(x) \subseteq cl(x) \cap Ker(x) = cl_\lambda(x)$. On the other hand, since every λ-open set is the union of an open set and a saturated set, then $cl_\lambda(x) \subset A$ for every λ-open set A containing x. From this and corollary 3.9, every topological space X is λ-$\mathbf{R}_0$, a fact that was unnoticed by the authors of [3].

4 Relations, saturated sets and separation axioms

In this section we will introduce the notion of a saturated set with respect to a binary relation (like $k(\Delta)$, L_g and E_g). We will show that the results of the previous section can be stated in terms of algebraic properties of the collection of saturated sets.

Let E be a binary relation on a set X (i.e. $E \subseteq X \times X$). We will always assume that E contains the diagonal Δ. We say that a subset $A \subseteq X$ is *E-saturated* if whenever $x \in A$ and $(y, x) \in E$, then $y \in A$. The family of E-saturated sets will be denoted by $S[E]$.

The following result shows that the notion of an E-saturated set is a natural generalization of a $\mathbf{g}$-saturated set.

Proposition 4.1 *(i) $A \in S[L_g]$ iff for each $x \in A$, $k(x) \subseteq A$, i.e. $S(X) = S[L_g]$.*

(ii) $A \in S[E_g]$ iff $k_\lambda(x) \subseteq A$, for each $x \in A$.

(iii) $A \in S[k(\Delta)]$ iff $k_\theta(x) \subseteq A$, for each $x \in A$.

Proof. The proof follows from the fact that $(y, x) \in L_g$ iff $y \in k(x)$, $(y, x) \in E_g$ iff $k(x) = k(y)$ iff $y \in k_\lambda(x) = k(x) \cap \chi(x)$, and $(y, x) \in k(\Delta)$ iff $y \in k_\theta(x)$. ■

We show below a general fact about saturated sets which will be used several times in the sequel.

Lemma 4.2 *Let E be a binary relation over X.*

(i) $S[E]$ is closed under arbitrary unions and intersection.

(ii) If E is a symmetric relation, then $S[E]$ is a complete atomic Boolean algebra. Moreover, $S[E] = S[F]$ where F is the smallest equivalence relation containing E and the F-equivalence classes are the atoms of $S[E]$.

(iii) $S[E] = \mathcal{P}(X)$ *iff* $E = \Delta$.

Proof. (i) is obvious. (ii) To get the result it is enough to prove that $S[E]$ is closed under complements. Let $A \in S[E]$. If $X \backslash A \notin S[E]$, there exists $x, y \in X$ such that $y \in A$ and $(y, x) \in E$ but $x \notin A$. From the symmetry of E, it follows that $(x, y) \in E$ which implies that $x \in A$, a contradiction. Let F be the transitive closure of E, that is to say, $(x, y) \in F$ if there are $x_i \in X$, $i = 0, \cdots, n$ such that $x = x_0$, $y = x_n$ and $(x_i, x_{i+1}) \in E$. It is easy to check that F is the smallest equivalence relation containing E. Therefore $S[F] \subseteq S[E]$. On the other hand, it is routine to verify that each F-equivalence class $[x]_F$ is E-saturated. Moreover, if $z \in A \subseteq [x]_F$ and A is E-saturated, then $[z]_F = [x]_F$ and thus $A = [x]_F$. Hence the F-equivalence classes are the atoms of $S[E]$ and $S[E] = S[F]$.

(iii) One direction is obvious. For the other, suppose E is not equal to Δ and let $(x, y) \in E$ with $x \neq y$. Then $\{y\}$ is not E-saturated. ■

Remark 4.3 (*i*) Since $k(\Delta)$, $\chi(\Delta)$ and E_g are symmetric relations (lemma 3.2), then $S[k(\Delta)]$, $S[\chi(\Delta)]$ and $S[E_g]$ are complete atomic Boolean algebras. Now from theorem 3.6 and lemma 4.2, it follows immediately that $\mathbf{g}$ satisfies $(\mathbf{T}_2)$ iff $S[k(\Delta)] = \mathcal{P}(X)$ iff every cofinite set belongs to $S[k(\Delta)]$. Clearly the axioms T_1 and T_0 are characterized in an analogous way.

(*ii*) If $\mathbf{g}$ is a topology, $S[k(\Delta)]$ is denoted by $B_\theta(X)$ in [4]. It was proved there that $B_\theta(X)$ is complete Boolean algebra. Note that this result is an immediate consequence of lemma 4.2(ii).

Our next results deal with the axioms $(\mathbf{R}_0)$ and $(\mathbf{R}_1)$.

Theorem 4.4 *The following are equivalent.*

(*i*) $\mathbf{g}$ *satisfies* $(\mathbf{R}_0)$.

(*ii*) $S[L_g]$ *is a complete atomic Boolean algebra.*

(*iii*) $\mathbf{g} \subseteq S[L_g]$.

Proof. The equivalence $(i) \leftrightarrow (iii)$ was proved in [5] lemma 3.2. It is clear that $\mathbf{g}$ satisfies $(\mathbf{R}_0)$ iff L_g is a symmetric relation. Therefore $(i) \to (ii)$ follows from lemma 4.2(ii). For the reverse implication, note that if $x \in X$ and $z \in k(x)$, then $k(z) \subseteq k(k(x)) = k(x)$, thus $k(x) \in S(X)$. Suppose $S[L_g]$ is a complete Boolean algebra, and let $y \in X$ and $x \in k(y)$. If $y \notin k(x)$, then $y \in X \backslash k(x) \in S(X)$ and we will have that $x \in k(y) \subseteq X \backslash k(x)$, a contradiction. Thus $y \in k(x)$ which shows that $(ii) \to (i)$. ■

Theorem 4.5 *The following are equivalent.*

(i) $\mathbf{g}$ *satisfies* $(\mathbf{R}_1)$.

(ii) $S[k(\Delta)]$ *is a complete atomic Boolean algebra and the sets* $k(x)$ *(*$x \in X$*) are its atoms.*

(iii) $\mathbf{g} \subseteq S[k(\Delta)]$.

Proof. $(i) \rightarrow (ii)$. Suppose $\mathbf{g}$ satisfies $(\mathbf{R}_1)$. Since $k(\Delta)$ is symmetric, then by lemma 4.2 $S[k(\Delta)]$ is a complete atomic Boolean algebra. Since $k(\Delta) = L_g$ (theorem 3.8), then each $k(x)$ is $k(\Delta)$-saturated. To show that the sets $k(x)$ are the atoms, let $z \in A \subseteq k(x)$ with A a $k(\Delta)$-saturated set. Then $z \in k(x)$ and thus $(z, x) \in k(\Delta)$. By symmetry $(x, z) \in k(\Delta)$ and as A is $k(\Delta)$-saturated, then $x \in A$. Hence $A = k(x)$.

$(ii) \rightarrow (iii)$. Suppose (ii) holds. We will show that $S[L_g] = S[k(\Delta)]$ and the result will follow from theorem 4.4. Since $L_g \subseteq k(L_g) = k(\Delta)$ (lemma 3.4), then $S[k(\Delta)] \subseteq S[L_g]$. Conversely, if A is L_g-saturated, then A is equal to the union of the sets $k(x)$ with $x \in A$. But by hypothesis each $k(x)$ belongs to the complete algebra $S[k(\Delta)]$, thus $A \in S[k(\Delta)]$.

$(iii) \rightarrow (i)$. Suppose $\mathbf{g} \subseteq S[k(\Delta)]$. We will show that $k(\Delta) = L_g$, and from this and theorem 3.8 the result follows. Let $(x, y) \in k(\Delta)$. Given $A \in \mathbf{g}$ with $y \in A$, then $A \in S[k(\Delta)]$ and thus $x \in A$. Then $x \in k(y)$ and therefore $(x, y) \in L_g$. Since $L_g \subseteq k(L_g) = k(\Delta)$, we conclude that $k(\Delta) = L_g$. ■

5 $T_{1/2}$ and $T_{1/4}$

In this section we characterize the axioms $T_{1/2}$ and $T_{1/4}$ in terms of properties of the diagonal and also in terms of properties of the family of saturated sets. We start with a general result about envelopes.

Lemma 5.1 *Let* $\mathbf{g}$ *be a generalized topology on* X *and let* ρ *be an envelope such that* $\rho(x) = k(x)$ *for all* $x \in X$*. For each* $A \subseteq X$*, the following are equivalent:*

(i) $A = \rho(A) \cap \chi(A)$.

(ii) $\rho(x) \subseteq \rho(A) \setminus A$*, for all* $x \in \rho(A) \setminus A$.

Proof. $(i) \rightarrow (ii)$. Suppose $A = \rho(A) \cap \chi(A)$ and let $x \in \rho(A) \setminus A$. Then $x \notin \chi(A)$ and thus there exists $H \in \mathbf{g}$ such that $A \subseteq H$ and $x \notin H$. Let

$y \in \rho(x) \subset \rho(A)$. If $y \in A$, then $y \in H$ and it must be that $x \in H$ since $y \in k(x) = \rho(x)$, a contradiction. Thus $y \notin A$ and therefore $\rho(x) \subseteq \rho(A) \setminus A$.

$(ii) \rightarrow (i)$. Conversely, suppose $\rho(x) \subseteq \rho(A) \setminus A$, for all $x \in \rho(A) \setminus A$. Let $z \in \rho(A) \cap \chi(A)$. If $z \notin A$, then $\rho(z) \subseteq \rho(A) \setminus A$ and it is clear that $A \subset X \setminus \rho(z)$. Since $\rho(z) = k(z)$, $z \in \chi(A)$ and $X \setminus k(z) \in \mathbf{g}$ (proposition 2.4), then it must be that $z \in X \setminus k(z)$, a contradiction. Therefore $A = \rho(A) \cap \chi(A)$. ∎

Recall from section 2 the definition of the envelope operations $k_\lambda(A) = k(A) \cap \chi(A)$ and $k_\mu(A) = sat(A) \cap \chi(A)$, $A \subset X$. We denote $A' = k(A) \setminus A$ and $A^* = sat(A) \setminus A$. The following result is an immediate consequence of lemma 5.1.

Corollary 5.2 *Let $A \subseteq X$. Then*

(i) $A = k_\lambda(A)$ *iff* $A' \in S(X)$.

(ii) $A = k_\mu(A)$ *iff* $A^* \in S(X)$.

It is known that a topological space X is $T_{1/2}$ iff every subset of X is λ-closed [1]. This result inspired part of the theorems 5.3 and 5.4 that follows.

Theorem 5.3 *The following are equivalent.*

(i) $\mathbf{g}$ *satisfies* $(T_{1/4})$.

(ii) $A = k_\mu(A)$ *for all* $A \subset X$.

(iii) $A^* \in S(X)$ *for all* $A \subset X$.

(iv) *if* $sat(A) \subset \chi(A)$, *then* $sat(A) = A$.

Proof. The equivalence $(ii) \leftrightarrow (iii)$, follows from corollary 5.2(ii).

$(i) \leftrightarrow (ii)$. Suppose $\mathbf{g}$ satisfies $(T_{1/4})$ and let $A \subset X$. Let $A_1 = \{x \in X \setminus A : \chi(x) = \{x\}\}$ and $A_2 = X \setminus (A \cup A_1)$. Notice that A_1 is kerneled (by proposition 2.6) and A_2 is saturated (since $k(x) = \{x\}$ for every $x \in A_2$). Since $A = A_1^c \cap A_2^c$, then $sat(A) \subseteq A_1^c$ and $\chi(A) \subseteq A_2^c$. Therefore $A = sat(A) \cap \chi(A)$. Conversely, suppose that $A = sat(A) \cap \chi(A)$ for all $A \subset X$ and let $x \in X$. If $\{x\}$ is not kerneled, then $X \setminus \{x\}$ is not saturated. Since X is the only saturated set containing $X \setminus \{x\}$, this set must be kerneled, and thus $\{x\}$ is saturated.

$(iv) \leftrightarrow (i)$. Suppose (iv) holds and let $x \in X$. If $\{x\}$ is not kerneled, then $X \setminus \{x\}$ is not saturated. Hence there exists $y \in sat(X \setminus \{x\})$ such that $y \notin \chi(X \setminus \{x\})$, which implies that $X \setminus \{x\}$ is kerneled and therefore $\{x\}$ is closed. Conversely, suppose (i) holds and let $A \subset X$ such that $sat(A) \subset \chi(A)$. If A is not saturated, there is $x \in sat(A) \setminus A$. By hypothesis $\{x\}$ is kerneled or closed. If $\{x\}$ is kerneled, then $X \setminus \{x\}$ is a saturated set containing A, thus $X \setminus \{x\}$ contains $sat(A)$, a contradiction. If $\{x\}$ is closed, then $X \setminus \{x\}$ is kerneled and hence $X \setminus \{x\} \supset \chi(A) \supset sat(A)$, again a contradiction. Therefore $sat(A) = A$. ∎

By replacing the envelope sat by the envelope k in the proof of theorem 5.3, we obtain the following result.

Theorem 5.4 *The following are equivalent.*

(i) $\mathbf{g}$ *satisfies* $(T_{1/2})$.

(ii) $A = k_\lambda(A)$ *for all* $A \subset X$.

(iii) $A' \in S(X)$ *for all* $A \subset X$.

(iv) *For all* $A \subset X$, *if* $k(A) \subset \chi(A)$, *then* $k(A) = A$.

The following two results show that the axioms $(T_{1/2})$ and $(T_{1/4})$ can also be characterized in terms of properties of Δ.

Theorem 5.5 $\mathbf{g}$ *satisfies* $(T_{1/2})$ *iff* $\Delta = \Delta_1 \cup \Delta_2$, *where* $\Delta_1 \in \mathbf{g}^2$ *and* Δ_2 *is saturated.*

Proof. $(\Rightarrow)$ Suppose $\mathbf{g}$ satisfies $(T_{1/2})$. Let $A_1 = \{x \in X : \{x\} \in \mathbf{g}\}$ and $A_2 = \{x \in X : k(x) = \{x\}\}$, and let $\Delta_i = \bigcup_{x \in A_i} \{x\} \times \{x\}$, $i = 1, 2$. By definition of $(T_{1/2})$, it is obvious that $\Delta = \Delta_1 \cup \Delta_2$. Also, $\Delta_1 = \bigcup_{x \in A_1} \{x\} \times \{x\} \in \mathbf{g}^2$ and $sat(\Delta_2) = \bigcup_{x \in A_2} k(x) \times k(x) = \bigcup_{x \in A_1} \{x\} \times \{x\} = \Delta_2$.

$(\Leftarrow)$ Suppose $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1 \in \mathbf{g}^2$ and Δ_2 is saturated. Since $\Delta_1 \subset \Delta$, there exists a set $B_1 \subset X$ such that $\Delta_1 = \bigcup_{x \in B_1} \{x\} \times \{x\}$. Also, $\Delta_1 \in \mathbf{g}^2$ implies that $\Delta_1 = \bigcup_\alpha A_\alpha \times B_\alpha$, with $A_\alpha, B_\alpha \in \mathbf{g}$. Thus, for each $x \in B_1$, $\{x\} \times \{x\} = A_\alpha \times B_\alpha$ for some α, and visceversa, and it follows that $\{x\} \in \mathbf{g}$ for each $x \in B_1$. On the other hand, $\Delta_2 = sat(\Delta_2) = \bigcup_{x \in B_2} k(x) \times k(x) \subset \Delta$, for some set $B_2 \subset X$, and it follows that $k(x) = \{x\}$, for each $x \in B_2$. It is clear that $X = B_1 \cup B_2$. Therefore $\mathbf{g}$ satisfies $(T_{1/2})$. ∎

Theorem 5.6 $\mathbf{g}$ *satisfies* $(T_{1/4})$ *iff* $\Delta = \Delta_1 \cup \Delta_2$, *where* Δ_1 *is kerneled and* Δ_2 *is saturated.*

Proof. ($\Rightarrow$) Suppose g satisfies $(T_{1/4})$. Let $A_1 = \{x \in X : \chi(x) = \{x\}\}$ and $A_2 = \{x \in X : k(x) = \{x\}\}$ and let $\Delta_i = \bigcup_{x \in A_i} \{x\} \times \{x\}$. It is clear that $\Delta = \Delta_1 \cup \Delta_2$, $\chi(\Delta_1) = \Delta_1$, and $sat(\Delta_2) = \Delta_2$.

($\Leftarrow$) Suppose $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 is kerneled and Δ_2 is saturated. There exists $B_1 \subset X$ such that $\Delta_1 = \bigcup_{x \in B_2} \chi(x) \times \chi(x)$. Since $\Delta_1 \subset \Delta$, it follows that $\chi(x) = \{x\}$ for each $x \in B_1$. Also, there exist $B_2 \subset X$ such that $k(x) = \{x\}$, for each $x \in B_2$, and since $X = B_1 \cup B_2$, we conclude that **g** satisfies $(T_{1/4})$. $\blacksquare$

Some results found in [2] and [3] can be obtained directly from those proved here, by considering the generalized topologies given by the α-open sets and the λ-open sets respectively.

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