Interpolation of sequences

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January 4, 2006

Abstract

We present a generalization of the following result of Y. Benyamini: There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for each $(x_n)_{n\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$, there is $t \in \mathbb{R}$ such that $x_n = f(t+n)$ for all $n \in \mathbb{Z}$.

Keywords: Interpolation of sequences, Universal surjectivity of the Cantor set.
AMS Subject Class 2000: Primary 54E45; Secondary 54H05.

1 Introduction

As an example of an universal property of the Cantor set, Y. Benyamini has shown [1] that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for any sequence $(x_n)_{n\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$, there is $t \in \mathbb{R}$ such that $x_n = f(t+n)$ for all $n \in \mathbb{Z}$. Such function f is said to interpolate all sequences in $[0,1]^{\mathbb{Z}}$. He also showed that it is not possible to interpolate all bounded $\mathbb{Z}$-sequences with a single continuous function (in fact, it is not possible to interpolate all constant $\mathbb{Z}$-sequences). In contrast with the last result, he showed that it is possible to interpolate all bounded sequences in $\mathbb{R}^{\mathbb{N}}$. In this note we will continue this line of investigation and present some extensions of those results.

In order to state our results we need to introduce some notations and recall some notions. An ideal over a set X is a collection $\mathcal{I}$ of subsets of X such that: (i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$; (ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal is said to be a σ-ideal, if it is closed under countable unions. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a subset $M \subseteq \mathbb{Z}$, we define a set of M-sequences as follows:

$$S_M(f) = \{(f(t+n))_{n\in M} :\ t \in \mathbb{R}\}.$$

When $M = \mathbb{Z}$ we will just write $S(f)$ instead of $S_{\mathbb{Z}}(f)$. We say that a set $S \subseteq \mathbb{R}^M$ is interpolated by f if $S \subseteq S_M(f)$. Consider the following family

$$\mathcal{C}(M) = \{S \subseteq \mathbb{R}^M :\ S \subseteq S_M(f) \text{ for some continuous } f : \mathbb{R} \to \mathbb{R}\}.$$

For each $M \subseteq \mathbb{Z}$, we define

$$\mathsf{h}(M) = \sup\{b - a :\ a, b \in \mathbb{Z} \text{ and } M \cap [a,b] = \emptyset\}.$$

Notice that $\mathsf{h}(M)$ measures the size of the "largest hole" of M inside $\mathbb{Z}$.

Our main result is the following:

Theorem 1.1. *Let $M \subseteq \mathbb{Z}$.*

(i) $\mathcal{C}(M)$ is an ideal of subsets of $\mathbb{R}^M$ containing every compact subset of $\mathbb{R}^M$.

(ii) $\mathcal{C}(M)$ is the σ-ideal generated by the compact subsets of $\mathbb{R}^M$ iff $h(M) = +\infty$.

That every compact set belongs to $\mathcal{C}(M)$ was essentially proved in [1] and it is a consequence of the Alexandroff-Hausdorff's theorem which says that every compact metric space is the continuous image of the Cantor set. As we said before, in [1] was also shown that the collection of bounded sequences in $\mathbb{R}^{\mathbb{N}}$ (i.e. the set $\bigcup_{n\in\mathbb{N}}[-n,n]^{\mathbb{N}}$) belongs to $\mathcal{C}(\mathbb{N})$. This follows from 1.1(ii) as $\mathsf{h}(\mathbb{N}) = +\infty$.

The next theorem shows another extension of Benyamini's result.

Theorem 1.2. *(i) There is a continuous function $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ such that $S(f) = \mathbb{R}^{\mathbb{Z}}$.*

(ii) There is a Baire class-1 function $f : \mathbb{R} \to \mathbb{R}$ such that $S(f) = \mathbb{R}^{\mathbb{Z}}$.

2 Proofs of the main results

For a given $K \subseteq \mathbb{R}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, we define

$$K + m = \{(x_{n+m})_{n\in\mathbb{Z}} : \ (x_n)_{n\in\mathbb{Z}} \in K\}.$$

Part of the following result is a convenient restatement of a result from [1].

Lemma 2.1. *For every $f : \mathbb{R} \to \mathbb{R}$ continuous there is a compact $K \subseteq \mathbb{R}^{\mathbb{Z}}$ such that $S(f) = \bigcup_{n\in\mathbb{Z}} K + n$. Conversely, for every compact $K \subseteq \mathbb{R}^{\mathbb{Z}}$ there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\bigcup_{n\in\mathbb{Z}} K + n \subseteq S(f)$.*

Proof. Let K be the set of all sequences $(f(t+n))_{n\in\mathbb{Z}}$ with $t \in [0,1]$. Then K is clearly a compact subset of $\mathbb{R}^{\mathbb{Z}}$. We claim that $S(f) = \bigcup_{n\in\mathbb{Z}} K + n$. In fact, given $t \in \mathbb{R}$, let $x_n = f(t+n)$ for $n \in \mathbb{Z}$. Note that $f(t+n) = f(t - \lfloor t \rfloor + n + \lfloor t \rfloor)$. Hence $(x_n)_{n\in\mathbb{Z}} \in K + \lfloor t \rfloor$. On the other hand, $K \subseteq S(f)$ and clearly $S(f) + n \subseteq S(f)$ for all $n \in \mathbb{Z}$.

Conversely, let $K \subseteq \mathbb{R}^{\mathbb{Z}}$ be a compact set. By the Alexandroff-Hausdorff theorem (see [2, 4.5.9] or [3, 4.18]), K is the continuous image of the Cantor set. Thus there is a Cantor set $\Delta \subseteq [0,1/2]$ and a continuous surjection $\phi : \Delta \to K$. Let $\tilde{f} : \bigcup_{n\in\mathbb{Z}} \Delta + n \to \mathbb{R}$ given by $\tilde{f}(t+n) = \phi(t)(n)$ for $t \in \Delta$ and $n \in \mathbb{Z}$. Since $\bigcup_{n\in\mathbb{Z}} \Delta + n$ is clearly closed, then by the Tietze's extension theorem $\tilde{f}$ can be extended to a continuous function $f : \mathbb{R} \to \mathbb{R}$. We claim that f works. In fact, given $(x_n)_n \in K$ there is $t \in \Delta$ such that $\phi(t) = (x_n)_n$. Thus $x_n = f(t+n)$ for all $n \in \mathbb{Z}$ and hence $K \subseteq S(f)$. Since $S(f) + n = S(f)$ for all $n \in \mathbb{Z}$, then $K + n \subseteq S(f)$ for all $n \in \mathbb{Z}$.

□

Our next result shows part (i) of theorem 1.1.

Lemma 2.2. *Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be continuous functions and $M \subseteq \mathbb{Z}$. Then there is $g : \mathbb{R} \to \mathbb{R}$ continuous such that $S_M(f_1) \cup S_M(f_2) \subseteq S_M(g)$. In particular, $\mathcal{C}(M)$ is an ideal of subsets of $\mathbb{R}^M$ containing every compact subset of $\mathbb{R}^M$.*

Proof. By lemma 2.1 there are compact sets K_i, for $i = 1, 2$, such that $S(f_i) = \bigcup_{n\in\mathbb{Z}} K_i + n$. Then $S(f_1) \cup S(f_2) \subseteq \bigcup_{n\in\mathbb{Z}}(K_1 \cup K_2) + n$. Since $K_1 \cup K_2$ is also compact, then by lemma 2.1 there is a continuous function g such that $S(f_1) \cup S(f_2) \subseteq S(g)$. Therefore $S_M(f_1) \cup S_M(f_2) \subseteq S_M(g)$. □

Now we show that if M has arbitrarily large "holes", then $\mathcal{C}(M)$ is closed under countable unions. To prove this we could repeat the argument as in the proof of lemma 2.1 (i.e. use the Alexandroff-Hausdroff's theorem), instead of this somewhat direct approach, for our proof we only need to know that $\mathcal{C}(M)$ contains every compact subset of $\mathbb{R}^M$.

Lemma 2.3. *Let $M \subseteq \mathbb{Z}$ be such that $\mathsf{h}(M) = +\infty$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a continuous function for each $n \in \mathbb{N}$. Then there is $g : \mathbb{R} \to \mathbb{R}$ continuous such that $\bigcup_n S_M(f_n) \subseteq S_M(g)$. In particular, $\mathcal{C}(M)$ is the σ-ideal generated by the compact subsets of $\mathbb{R}^M$.*

Proof. An argument similar to that used in the proof of lemma 2.1 easily shows that $S_M(f)$ is a countable union of compact sets. Therefore it suffices to show that $\bigcup_i K_i \in \mathcal{C}(M)$ if $K_i \subseteq \mathbb{R}^M$ is compact for each $i \in \mathbb{N}$. Fix then a sequence $(K_i)_i$ of compact subsets of $\mathbb{R}^M$. Then we can define for each i the following M-sequence

$$\alpha_i(n) = \sup\{|x_n| : \ (x_m)_{m\in M} \in K_i\},$$

for $n \in M$. To find a sort of an "uniform bound" for all K_i we use the fact that $h(M) = +\infty$. For each positive integer i, there is an integer k_i such that $(M - k_i) \cap [-i, i] = \emptyset$. Therefore $\{i \in \mathbb{N} : \ n \in M - k_i\}$ is finite for all $n \in \mathbb{Z}$. We fix such sequence $(k_i)_i$ and define, for each $n \in \mathbb{Z}$,

$$\alpha(n) = \max\{\alpha_i(n + k_i) : \ n + k_i \in M\}$$

and $\alpha(n) = 1$ if there is no i such that $n + k_i \in M$.

Now consider the compact set $K \subseteq \mathbb{R}^{\mathbb{Z}}$ given by $(x_n)_{n\in\mathbb{Z}} \in K$ iff $|x_n| \leq \alpha(n)$ for all $n \in \mathbb{Z}$. By lemma 2.1 there is a continuous function f such that $K \subseteq S(f)$. We claim that $K_i \subseteq S_M(f)$ for all i. In fact, fix i and let $(x_n)_{n\in M} \in K_i$. Define $(\tilde{x}_n)_{n\in\mathbb{Z}}$ by

$$\tilde{x}_n = \begin{cases} x_{n+k_i} & , \text{if } n + k_i \in M; \\ 0 & , \text{otherwise.} \end{cases}$$

We claim that $(\tilde{x}_n)_{n\in\mathbb{Z}} \in K$. In fact, let $n \in \mathbb{Z}$ be such that $n + k_i \in M$, then $|\tilde{x}_n| = |x_{n+k_i}| \leq \alpha_i(n + k_i) \leq \alpha(n)$ and we are done. On the other hand, since $K \subseteq S(f)$, there is $t \in \mathbb{R}$ such that $\tilde{x}_n = f(t + n)$ for all $n \in \mathbb{Z}$. Finally, given $m \in M$, we have $x_m = \tilde{x}_{m-k_i} = f(t + m - k_i)$. Thus $(x_m)_{m\in M} \in S_M(f)$. □

To complete the proof of Theorem 1.1, it remains only to show that the hypothesis about $\mathsf{h}(M)$ is necessary in the previous result.

Lemma 2.4. *Let $M \subseteq \mathbb{Z}$ such that $\mathsf{h}(M) < +\infty$. Then there is no continuous function g such that $\bigcup_{n=1}^{\infty}[0, n]^M \subseteq S_M(g)$.*

Proof. Suppose that $\mathsf{h}(M) = q$. Towards a contradiction, suppose there is such a function g. For each $n \in \mathbb{N}$, consider the constant M-sequence equal to n. Then there is $t_n \in \mathbb{R}$ such that $f(t_n + m) = n$ for all $m \in M$. Since $\mathsf{h}(M) = q$, then $(M + t_n) \cap [0, q + 1] \neq \emptyset$ for all $n \in \mathbb{N}$. That is to say, for all $n \in \mathbb{N}$, there is $s_n \in [0, q+1]$ such that $g(s_n) = n$. This is impossible, as g is continuous and $[0, q+1]$ is compact.

$\square$

Proof of Theorem 1.2: It is well known that the set of irrational numbers in homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$. It is a classical result that every Polish space (i.e. a separable and completely metrizable space) is the continuous image of $\mathbb{N}^{\mathbb{N}}$ (see for instance [3, Theorem 7.9]).

(i) Since $\mathbb{R}^{\mathbb{Z}}$ is a Polish space, by the result mentioned above, there is a continuous surjection $\phi : (0,1) \setminus \mathbb{Q} \to \mathbb{R}^{\mathbb{Z}}$. Define $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ by $f(t+n) = \phi(t)(n)$ for $t \in (0,1) \setminus \mathbb{Q}$ and $n \in \mathbb{Z}$. It is routine to verify that such f works.

(ii) Another classical result says that there is a Baire class-1 surjection $h : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ (see [4, 1G.10, pag. 58]). Hence there is a Baire class-1 surjection $\phi : \Delta \to \mathbb{R}^M$ where $\Delta \subseteq [0, 1/2]$ is a Cantor set. From this point on the argument is analogous to that used in the proof of lemma 2.1 (it is easy to verify that if $A \subseteq \mathbb{R}$ is closed and $f : A \to \mathbb{R}$ is a Baire class-1 function, then f can be extended to a Baire class-1 function defined on $\mathbb{R}$).

$\square$

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