Maximal complements in the lattices of pre-orders and topologies

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Abstract

Two topologies \( \tau \) and \( \rho \) over \( X \) are said to be complementary if \( \tau \land \rho \) is the indiscrete topology and \( \tau \lor \rho \) the discrete topology. The lattice of topologies is complemented, i.e, every topology has a complement. We will show that every AT topology (i.e. a topology such that the intersection of arbitrary many open sets is open) over a countable set has a maximal complement in the lattice of topologies. This result answers a question of S. Watson (Top. and Appl.. 55(1994), 101-125). This theorem is a corollary of an analogous result for the lattice of pre-orders. We show that every pre-order \( P \) on a countable set \( X \) admits a maximal complement in the lattice of preorders over \( X \). Moreover, if every connected component of \( P \) is neither discrete nor indiscrete, then such a maximal complement has all its chains of size at most two.

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1 Introduction

The collection of topologies \( \text{TOP}(X) \) over a set \( X \) is a lattice under inclusion \( \subseteq \). The greatest element is the discrete topology (where every set is open) and the smallest element is the indiscrete topology (whose open sets are just \( \emptyset \) and \( X \)). The lattice operations are defined by letting the meet \( \tau \land \rho \) of two topologies be \( \tau \cap \rho \) and the join \( \tau \lor \rho \) be the least topology which contains both \( \tau \) and \( \rho \) (i.e., the topology having \( \tau \cup \rho \) as a subbasis). Moreover, \( \text{TOP}(X) \) is a complete lattice. Two topologies \( \tau \) and \( \rho \) over \( X \) are said to be complementary if \( \tau \land \rho \) is the indiscrete topology and \( \tau \lor \rho \) the discrete topology. Steiner [1] has shown a long time ago that the lattice of topologies is complemented, i.e, every topology has a complement. We refer the reader to Watson’s paper [3] where many new results about the complementation in \( \text{TOP}(X) \) and an extensive bibliography can be found. This paper is motivated by a question from [3]. It was asked whether there are topologies with a maximal complement. It is known that a non discrete \( T_1 \) topology cannot have a maximal complement [2]. A topology is said to be an Alexandroff-Tucker (AT) topology if the intersection of arbitrary many open sets is open. AT topologies are non \( T_1 \) (except for the discrete topology). We will show the following

**Theorem 1:** Every AT topology over a countable set has a maximal complement in the lattice of topologies.
This theorem is a corollary of an analogous result for the lattice of pre-orders. This lattice is tightly connected to the lattice of topologies and especially to the complementation of topologies. Let us recall some known facts about them. The collection $PO(X)$ of pre-orders over a set $X$ (i.e. transitive and reflexive binary relations but not necessarily antisymmetric) also forms a lattice under reversed inclusion. The join of two pre-orders $P$ and $Q$ is then $P \cap Q$ and their meet is the transitive closure of $P \cup Q$. Moreover $PO(X)$ is also a complete complemented lattice. Complementation in $PO(X)$ is quite natural. Two pre-orders $P$ and $Q$ are complementary if their intersection, as binary relations, is the identity relation (denoted by $\Delta$) and the transitive closure of their union is the largest binary relation $X \times X$. The reader should keep in mind that a maximal complement in $PO(X)$ is in fact $\subseteq$-minimal, since in $PO(X)$ the lattice order is given by reversed inclusion. Our main result is the following

**Theorem 2:** Every pre-order $P$ on a countable set $X$ admits a maximal complement in $PO(X)$.

Moreover, if every connected component of $P$ is neither discrete nor indiscrete, then a maximal complement can be found with all its chains of size at most two. The following diagram shows a maximal complement for $Z$ with its usual order. The pairs labelled with $\nearrow$ form the complementary relation.

\[
\begin{array}{cccccccc}
-5 & -3 & -1 & 1 & 3 & 5 & 7 \\
\cdots & \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \cdots \\
-6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\end{array}
\]

Observe that all pairs labelled with $\nearrow$ belong to the usual order in $Z$. Notice that if we erase one single $\nearrow$, then the resulting relation $Q$ is not a complement of $Z$, because the pair removed cannot be recovered by the transitive closure of $Q$ together with the order of $Z$.

Let $\preceq$ be a pre-order over a set $X$, its associate AT topology $\tau_{\preceq}$ is defined as the topology generated by the collection of sets of the form $\{x \in X : a \preceq x\}$ with $a \in X$. This is in fact a characterization of AT topologies. Namely, given an AT topology $\tau$, define $\preceq_\tau$ by $x \preceq_\tau y$ iff $x \in \{y\}$. Then $\preceq_\tau$ is a pre-order and $\tau$ is its associated AT topology. Moreover, $\tau$ is $T_0$ iff $\preceq_\tau$ is a partial order (i.e it is antisymmetric). The lattice order in $PO(X)$ is taken as reversed inclusion so that it coincides with the induced order when we view $PO(X)$ (via $\tau_{\preceq}$) as a subset of $TOP(X)$. However $PO(X)$ is not a sublattice of $TOP(X)$.

Thus the question about the existence of maximal complements for an AT topology has two natural variants according to where we look at it: either inside $PO(X)$ or inside $TOP(X)$. However, if $\rho$ is an AT topology, then the topology generated by $\rho$ together with finitely many sets is again AT. Therefore if a complement of an AT topology is maximal in $PO(X)$, then it is also maximal in $TOP(X)$. In other words, Theorem 1 follows from Theorem 2. On the other hand, we have some partial results suggesting that some subtle conditions must be imposed in order to answer the general question about the existence of a maximal complement for an arbitrary topology over a countable set.

Our result cannot be extended to arbitrary pre-orders over an uncountable set. Since, as we will show, $\omega_1$ with its usual order does not have a maximal complement in $PO(\omega_1)$. On the other hand, the main result can be extended to any pre-order $P$ such that both $P$ and $P^{-1}$ are separable. In fact, the construction can be carried out inside a countable dense and co-dense subset of $X$. Therefore these type of partial orders do have maximal complements.

We will make next some comments about the analogous question for minimal complements. A standard Zorn’s lemma argument shows that given a pre-order $P$ and a complement $Q$ of $P$ there is a $\subseteq$-maximal partial order $R$ extending $Q$ and such that $R \cap P = \Delta$. It is clear that such $R$ is
a minimal complement of $P$ in $PO(X)$. Thus the existence of minimal complement in $PO(X)$ is quite easy to establish. However, it is not clear if such minimal complements are also minimal in $TOP(X)$. On the other hand, nothing similar happens for maximal complements, that is to say, there is a partial order $P$ and a complement $Q$ of $P$ such that there is no maximal complement of $P$ above $Q$ in $PO(X)$. In fact, consider the following example. Let $P$ be $\mathbb{N}$ with its usual order and $Q_A = \{(n, 0): n \in A\} \cup \Delta$ for $A \subseteq \mathbb{N}$. Notice $Q_A$ is a complement of $P$ iff $A$ is infinite. Therefore for every infinite $A$ there is no $R \subseteq Q_A$ such that $R$ is a maximal complement for $P$.

We end the introduction by fixing the notation and terminology. $X$ will always denote a countable set. We will denote a pre-order over $X$ either by $\preceq$ or just $P$ as a binary relation. In case we have more than one pre-order we will write $\preceq_P$ to avoid any possible confusion. The strict relation is denoted by $\prec$, that is to say, $x \prec y$ if $x \leq y$ and $y \neq x$. Since a pre-order is not necessarily antisymmetric it is convenient to use the following equivalence relation: let $x \sim y$ if $x \leq y$ and $y \leq x$. A subset $Y$ of a pre-ordered set $(X, \preceq)$ is said to be open (in the associated AT topology of $(X, \preceq)$) if whenever $x \in Y$ and $x \preceq y$, then $y \in Y$. An element $x$ of a pre-ordered set $(X, \preceq)$ will be called clopen if whenever $y \preceq x \preceq z$ for some $y, z \in X$, then $x \sim y \sim z$ (i.e. the $\sim$-equivalence class of $x$ is clopen). A non clopen element $x$ of $X$ will be called maximal if there is no $y \in X$ such that $x \prec y$. The notion of a minimal element is defined analogously. The collections of maximal and minimal elements of $X$ will be respectively denoted by $Max$ and $Min$. A subset $A \subseteq X$ is said to be up-dense if for every $x \in X$ there is $y \in X$ such that $x \preceq y$. Analogously we define the notion of a down-dense set. A (connected) component of a pre-order $P$ is a non-empty subset $D$ of $X$ such that for every $x, y \in D$, there is a path in $P \cup P^{-1}$ from $x$ to $y$. A component $D$ of $P$ is said to be trivial, if either $D$ has only one element or the restriction of $P$ to $D$ is equal to $D \times D$.

2 Maximal complements in the lattice of pre-orders

Our main result was suggested by the following example. Let $P$ be the partial order of all binary sequences $2^{\leq \omega}$ with the usual extension order $\preceq$. It is clear that there are elements $a_n, b_n$ in $2^{\leq \omega}$ such that $a_n \prec b_n$ and moreover the $a_n$’s form a dense set in $2^{\leq \omega}$. For instance, take the $a_n$’s to be the collection of all sequences of even length ordered by size and let $b_n$ be $a_n$ followed by $0$. Consider the following arrangement

\[
\begin{array}{cccccccccccc}
00 & 01 & 10 & 11 & 0000 & 0010 & 0010 & 00110 \\
\emptyset & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \ldots \\
0 & 00 & 01 & 10 & 11 & 0000 & 0001 & 0010 & 0011
\end{array}
\]

Where $\emptyset$ is the empty sequence (which is the minimum of $P$). The top row corresponds to $b_1, b_2, \ldots$ and the bottom row to $\emptyset, a_1, a_2, \ldots$. Notice that all pairs labelled by $\uparrow$ belong to $P$. The collection $Q$ of all pairs labelled by $\nearrow$ together with the identity $\Delta$ is a complementary relation for $P$. In fact, it is clear that $Q \cap P$ is equal to $\Delta$. To see that the transitive closure of $Q \cup P$ contains every element of $2^{\leq \omega}$, just observe that we can travel from any sequence to some $a_n$ (by the density of the $a_n$’s) and then move towards $\emptyset$ following the diagram above by using alternatively $P$ or $Q$. Moreover, $Q$ is a maximal complement for $P$.

For the rest of this section, $P$ will denote a pre-order over $X$. We will assume (unless stated otherwise) that $P$ has no trivial components (see the introduction). The general case will be treated at the end of this section.

The main ingredient of the proof will be finding sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ in $X$.
carefully arranged as shown in the diagram below, as was done for \( \mathbb{Z} \) and \( 2^{\omega} \).

\[
\begin{array}{cccccccc}
\cdots & d_3 & d_2 & d_1 & b_1 & b_2 & b_3 & b_4 & \cdots \\
& \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\
c_3 & c_2 & c_1 & a_1 & a_2 & a_3 & a_4 & & \\
\end{array}
\]

where every pair labelled with \( \uparrow \) belongs to \( P \) and every pair labelled with \( \nearrow \) does not belong to \( P \). Let \( Q \) be the collection of pairs labelled with \( \nearrow \). If the sequence \( \{a_n\} \) is up-dense in \( X \) and \( \{d_n\} \) is down-dense, then \( Q \) is a complement of \( P \). To make it maximal we will impose some extra conditions. The sequences \( \{a_n\} \) and \( \{b_n\} \) can indeed be found when \( X \) does not have maximal elements and analogously, when \( X \) does not have minimal elements, we can find the sequences \( \{c_n\} \) and \( \{d_n\} \). Thus the final step in the proof will be to partition \( X \) into four pieces such that in each piece we can find a maximal complement and then glue them together to get a maximal complement for the whole space.

The proof of theorem 1 and 2 will be given in a sequence of lemmas. The definition of \( \{a_n\} \), \( \{b_n\} \), \( \{c_n\} \) and \( \{d_n\} \) will be by recursion. The basic fact used in the inductive step is the following

**Lemma 2.1.** Let \( F \subseteq X \) be a finite set and \( x \in X \). Suppose \( P \) does not have maximal elements. Then

\[
\bigcap_{z \in F} \{y \in X : y \not\leq z\} \cap \{y \in X : x \leq y\} \neq \emptyset
\]

And dually (by reversing \( \leq \)), if \( X \) does not have minimal elements, then

\[
\bigcap_{z \in F} \{y \in X : z \not\geq y\} \cap \{y \in X : y \leq x\} \neq \emptyset
\]

**Proof:** Just notice that, when there are no maximal elements, the set \( \{y \in X : y \leq z\} \) is nowhere dense in \( X \) with the associated AT topology (see the introduction).

**Lemma 2.2.** Suppose \( P \) has neither maximal nor minimal elements. Then there are sequences \( \{a_n\}, \{b_n\}, \{c_n\} \) and \( \{d_n\} \) in \( X \) such that:

(i) For all \( x \in X \) there is \( n \) such that \( d_n \leq x \leq a_n \)

(ii) \( c_n \prec d_n \prec a_n \prec b_n \), for all \( n \).

(iii) \( a_n \not\leq b_m \) for all \( m < n \).

(iv) \( c_m \not\leq d_n \) for all \( m < n \).

Moreover, from (i), (ii), (iii) and (iv) we get the following

(v) \( a_n \not\leq d_m \) for all \( m \) and \( n \).

**Proof:** Let \( x_n \) be an enumeration of \( X \). The sequences \( \{a_n\}, \{b_n\}, \{c_n\} \) and \( \{d_n\} \) will be defined recursively. Since \( X \) has neither minimal nor maximal elements, pick \( a_1, b_1, c_1 \) and \( d_1 \) such that \( c_1 < d_1 < x_1 < a_1 < b_1 \).

Suppose we have defined \( a_i, b_i, c_i \) and \( d_i \) for \( i \leq k \) such that (ii)-(iv) holds and \( d_i \prec x_i \prec a_i \) for all \( i \leq k \). By lemma 2.1 there are \( a_{k+1} \) and \( d_{k+1} \) such that \( a_{k+1} \not\leq b_i \) for all \( i \leq k \), \( x_{k+1} \prec a_{k+1} \), \( c_i \not\leq d_{k+1} \) for all \( i \leq k \) and \( d_{k+1} \prec x_{k+1} \). Next, pick \( b_{k+1} \) and \( c_{k+1} \) such that \( a_{k+1} < b_{k+1} \) and \( c_{k+1} < d_{k+1} \).
It remains to be checked that condition (v) follows from (i)-(iv). By induction on \(k\) we show that \(a_k \not\leq d_i\) and \(a_i \not\leq d_k\) for all \(i \leq k\). In fact, from (ii) we have \(a_1 \not\leq d_1\). To see that \(a_{k+1} \not\leq d_i\) for \(i \leq k\) and \(a_i \not\leq d_{k+1}\) for \(i \leq k + 1\), notice that this follows from the fact that \(a_{k+1} \not\leq b_i\), \(c_i \not\leq d_{k+1}\) and \(c_i < d_i < a_i < b_i\) for all \(i \leq k\).

As we said before, if \(P\) has minimal elements but no maximal elements, then the previous construction can be obviously done to get only the sequences \(\{a_n\}\) and \(\{b_n\}\). By duality we have an analogous result when \(P\) has no minimal elements. We state these facts in the next lemma.

**Lemma 2.3.** (1) Suppose \(P\) does not have maximal elements. Then there are sequences \(\{a_n\}\) and \(\{b_n\}\) such that: (i) For all \(x \in X\) there is \(n\) such that \(x \leq a_n\); (ii) \(a_n \prec b_n\), for all \(n\) and (iii) \(a_n \not\leq b_m\) for all \(m < n\).

(2) Suppose \(P\) does not have minimal elements. Then there are sequences \(\{c_n\}\) and \(\{d_n\}\) such that: (i) For all \(x \in X\) there is \(n\) such that \(d_n \geq x\); (ii) \(c_n < d_n\), for all \(n\) and (iii) \(c_m \not\leq d_n\) for all \(m < n\).

**Lemma 2.4.** Suppose \(P\) has neither maximal nor minimal elements. Then \(P\) has a maximal complement in \(PO(X)\) that moreover has chains of size at most two.

**Proof:** Let \(a_n, b_n, c_n\) and \(d_n\) be as in lemma 2.2 and define an order \(Q\) (viewed as a binary relation) as follows

\[
Q = \{(b_{n+1}, a_n) : n \geq 1\} \cup \{(d_n, c_{n+1}) : n \geq 1\} \cup \{(b_1, c_1)\} \cup \Delta
\]

where \(\Delta\) is the diagonal.

First we show that \(Q\) is a complement of \(P\). From conditions (ii), (iii) and (iv) it is clear that \(P \cap Q = \Delta\). We will show that the transitive closure of \(P \cup Q\) is \(X \times X\). To avoid confusion, we will denote the pre-orders \(P\) and \(Q\) respectively by \(\leq_P\) and \(\leq_Q\). Let \(x, y \in X\) and let \(n, m\) be such that \(c_m \leq_Q y\) and \(x \leq_P b_n\). Then a path from \(x\) to \(y\) in \(P \cup Q\) is as follows

\[
x \leq_P b_n \leq_Q a_{n-1} \leq_P b_{n-1} \cdots \leq_Q a_1 \leq_P b_1 \leq_Q c_1 \leq_P d_1 \leq_Q \cdots \leq_Q c_m \leq_P y
\]

(1)

Next we show that \(Q\) is a maximal complement. It suffices to show that \(Q \setminus \{(x, y)\}\) is not a complement of \(P\) for any \((x, y) \in Q\) with \(x \neq y\). There are three cases for \((x, y)\) to consider:

- \((b_{n+1}, a_n)\), \((d_n, c_{n+1})\) or \((b_1, c_1)\). All three cases are similar.
- \((Case 1)\) Let \(Q' = Q \setminus \{(b_{n+1}, a_n)\}\). We will show that \((b_{n+1}, x)\) is not in the transitive closure of \(P \cup Q'\) when \(x\) is either \(c_m, d_m, a_i\) or \(b_i\) with \(i \leq n\). Suppose, towards a contradiction, that there is a path \(\alpha\) from \(b_{n+1}\) to \(x\) where \(x\) is either \(c_m, d_m, a_i\) or \(b_i\) for some \(i \leq n\). Assume that such \(\alpha\) has minimal length. It is clear that \(\alpha\) cannot have length 1, since it would be of the form \(b_{n+1} \leq_P x\) and from properties (ii) and (v) in lemma 2.2 we would get that such \(x\) can be equal neither to \(c_m\) nor to \(d_m\) for any \(m\), and from (ii) and (iii) we would get that \(x\) can be equal neither to \(a_i\) nor to \(b_i\) for \(i \leq n\). So we have that \(\alpha\) looks as follows

\[
b_{n+1} \leq_{R_1} x_1 \leq_{R_2} x_2 \cdots \leq_{R_k} x_k \leq_{R_{k+1}} x_{k+1}
\]

where \(k \geq 1\) and \(R_i\) is (alternately) either \(P\) or \(Q'\). By the minimality of the length of \(\alpha\) we necessarily have that \(x_k\) cannot be equal to neither \(c_m, d_m, a_i\) nor \(b_i\) with \(i \leq n\). Therefore \(x_k\) has to be equal to either \(a_j\) or \(b_j\) for some \(j > n\). We consider two cases: (i) \(R_{k+1} = Q'\) and (ii) \(R_{k+1} = P\). For case (i), we have that \(x_k\) has to be equal to \(b_j\) for some \(j > n\). Since \((b_{n+1}, a_n) \notin Q'\),
then $j > n + 1$. Thus $x_{k+1}$ is equal to $a_{j-1}$, but this contradicts our assumption as $j - 1 > n$. For case (ii), by the inductive hypothesis we can also suppose that $R_k = Q'$. Thus $x_k$ is equal to $a_j$ for some $j > n$. It follows from condition (ii) and (v) that $x_{k+1}$ can be equal neither to $c_i$ nor to $d_i$ for any $i$. Also, from condition (ii) and (iii) we get that $x_{k+1}$ can be equal neither to $a_i$ nor to $b_i$ for any $i < j$. Since $j > n$, we are done as before.

(Case 2) Let $Q'$ be $Q \setminus \{(d_n, c_{n+1})\}$. Analogously to case 1 it can be shown that $(d_n, x)$ is not in the transitive closure of $P \cup Q'$ when $x$ is either $c_m$ or $d_m$ for $m > n$.

(Case 3) Let $Q'$ be $Q \setminus \{(b_1, c_1)\}$. Analogously to case 1 it can be shown that $(b_1, x)$ is not in the transitive closure of $P \cup Q'$ when $x$ is either $c_m$ or $d_m$ for $m \geq 1$.

\[ \square \]

**Lemma 2.5.** Assume $P$ has no maximal elements and, in addition, suppose that every element of $X$ has at least one minimal element below it. Then $P$ has a maximal complement in PO$(X)$ that moreover has all its chains of size at most two.

**Proof:** Let $a_n$ and $b_n$ be as in lemma 2.3 and let $Min$ be the collection of minimal elements of $X$. Let $\sim$ be the equivalence relation over $X$ defined at the end of the introduction. Let $Min^*$ be formed by only one representative of each equivalent class of elements of $Min$. Define a partial order $Q$ as follows

\[ Q = \{(b_{n+1}, a_n) : n \geq 1\} \cup \{(b_1, z) : z \in Min^*\} \cup \Delta \]

where $\Delta$ is the diagonal. Since $a_1 \sim b_1$, then $b_1 \not\in Min$, therefore $P \cap Q = \Delta$. On the other hand, since any element $x$ in $X$ has an element of $Min$ below it, then it is clear that there is a path in $P \cup Q$ from $b_1$ to $x$. A path from $x$ to $b_1$ can be built as in (1). Thus $Q$ is a complement of $P$. To see that $Q$ is in fact maximal one has to consider two cases: (i) Let $Q_n$ be $Q \setminus \{(b_{n+1}, a_n)\}$, then $Q_n$ is not a complement of $P$. This is shown exactly as in the proof of lemma 2.4 by proving that $(b_{n+1}, x)$ is not in the transitive closure of $P \cup Q_n$ when $x$ is either $b_i$ with $i \leq n$ or any element of $Min^*$. (ii) For $z \in Min^*$, let $Q_z$ be $Q \setminus \{(b_1, z)\}$, then $Q_z$ is not a complement of $P$. In fact, it is easy to verify that $(b_1, z)$ is not in the transitive closure of $P \cup Q_z$, since whenever $(w, z) \in P$, then $w \in Min$.

\[ \square \]

A similar argument shows, mutatis mutandis, the following.

**Lemma 2.6.** Assume $P$ has no minimal elements and, in addition, suppose that every element has at least one maximal element above it. Then $P$ has a maximal complement $Q$ whose chains have at most two elements.

The next case we need to consider is when every element has at least one maximal element above it and also at least one minimal element below it. This case is handled in the following.

**Lemma 2.7.** Suppose that every element of $X$ has at least one maximal element above it and also at least one minimal element below it. Then $P$ has a maximal complement that moreover has all chains of size at most two.

**Proof:** Let $Min$ and $Max$ be respectively the collection of minimal and maximal elements of $X$. As in the proof of lemma 2.5, let $Min^*$ and $Max^*$ be formed by taking only one element of each equivalence class. Pick $a \in Min^*$ and $b \in Max^*$. Define $Q$ as follows

\[ \{(y, x) : x \in Min^* \& y \in Max^*\} \cup \Delta \]
It is straightforward to show that $Q$ is a maximal complement for $P$. \hfill $\square$

Now we will show how to glue together two maximal complements of a partition of the space and get a maximal complement of the whole space. This result is similar to proposition 2.10 of [3].

Recall that a subset $Y$ of $X$ is said to be open if for all $x, y$ with $x \leq y$ and $x \in Y$, then $y \in Y$.

**Lemma 2.8.** Let $X_1, X_2$ be a partition of $X$ such that $X_1$ is open. Let $P_i$ be the restriction of $P$ to $X_i$. Suppose that $Q_i$ is a maximal complement for $P_i$ with chains of at most two elements. Then $P$ has a maximal complement whose chains have at most two elements.

**Proof:** Since $X_1$ is open and $X_2$ is disjoint from $X_1$, then there is no $x \in X_1$ and $y \in X_2$ such that $x \leq y$. We consider two cases.

(i) $X_2$ is also open. Since each chain in $Q_i$ is of size at most two (and we are assuming that $P$ is not trivial) then there are $a \prec_{Q_1} b$ and $c \prec_{Q_2} d$. Define $Q$ as follows

$$Q = Q_1 \cup Q_2 \cup \{(d, a), (b, c)\}$$

Notice that since the $X_i$’s are both open, then $(d, a), (b, c) \not\in P$. Since each chain in $Q_i$ has at most 2 elements, then it is easy to verify that $Q$ is indeed transitive and moreover it is a pre-order whose chains has at most 2 elements. It is also routine to check that $Q$ is a complement for $P$. To see that $Q$ is a maximal complement, observe first that $Q_i$ is equal to the restriction of $Q$ to $X_i$. So it suffices to show that if we remove either $(d, a)$ or $(b, c)$ then the resulting pre-order is not a complement of $P$. But this is clear, since every path starting from a point of $X_1$ and ending in a point of $X_2$ necessarily uses $(b, c)$ and analogously every path starting from a point in $X_1$ and ending in a point of $X_2$ necessarily uses $(d, a)$. \hfill $\square$

(ii) Suppose now that $X_2$ is not open in $X$. As before, there are $a \prec_{Q_1} b$ and $c \prec_{Q_2} d$. Define $Q$ as follows

$$Q = Q_1 \cup Q_2 \cup \{(b, c)\}$$

Since $X_2$ is not open in $X$, then there are $x_i \in X_i$ such that $x_2 \prec x_1$. A completely similar argument as in case (i) but now replacing $(d, a)$ by $(x_2, x_1)$ shows that $Q$ is a maximal complement for $P$. \hfill $\square$

Now we have all we need to prove our main result. From this point on we will not assume that $P$ has no trivial components.

**Theorem 2.9.** Every pre-order $P$ over a countable set has a maximal complement in \(PO(X)\). Moreover, if $P$ has no trivial components a maximal complement can be found such that its chains have size at most two.

**Proof:** We first consider the case when $P$ has no trivial components. Let $Min$ and $Max$ be respectively the collection of minimal and maximal elements of $X$. Consider the following subsets of $X$:

- $X_1 = \{x \in X : \not\exists a \in Min \; \exists b \in Max \; a \leq x \leq b\}$
- $X_2 = \{x \in X : \exists a \in Min \; \exists b \in Max \; a \leq x \leq b\}$
- $X_3 = \{x \in X : \not\exists a \in Min \; \exists b \in Max \; a \leq x \leq b\}$
- $X_4 = \{x \in X : \exists a \in Min \; \exists b \in Max \; a \leq x \leq b\}$

It is clear that they form a partition of $X$ and moreover by lemmas 2.4, 2.5, 2.6 and 2.7 we know that $P$ restricted to each $X_i$ has a maximal complement with chains of size at most two. On the other hand, it is routine to verify the following
(i) $X_1 \cup X_2$ is open in $X$.
(ii) $X_2$ is open in $X_1 \cup X_2$.
(iii) $X_4$ is open in $X_3 \cup X_4$.

Therefore by lemma 2.8 and (ii) and (iii) above there are maximal complements for $P$ restricted to $X_1 \cup X_2$ and $X_3 \cup X_4$ with chains of size at most two. Now, again by lemma 2.8 and (i) above, we get a maximal complement for $P$.

Finally, we handle the trivial components. Let $Y$ be the union of all trivial components of $P$ and assume $Y$ is not empty. We consider two cases:

1. Suppose $Z = X \setminus Y$ is not empty. Notice that $Z$ and $Y$ are open. From the result we just proved, there is a maximal complement $Q$ for the restriction of $P$ to $Z$. Let $Y^*$ be a set containing one and only one element of each $\sim$-equivalence class of elements of $Y$. Pick $a, b \in Z$ such that $a \prec b$. Define a relation $R$ as follows:

$$R = Q \cup \{(a, y), (y, b) : y \in Y^*\}$$

It is routine to check that $R$ is a maximal complement for $P$. Notice that $R$ has a chain of size three.

2. Suppose $X = Y$ is empty. This is equivalent to saying that $P$ is an equivalence relation. We can assume that $P$ is neither equal to $\Delta$ nor to $X \times X$. Let $W$ be the set of all elements of $X$ whose equivalence class has size one and let $Z = X \setminus W$. We consider two cases.

2a. Suppose $Z$ has finitely many equivalence classes and let $\{a_n, b_n\}$ with $1 \leq n \leq m$ be a selection of two elements of each equivalence class in $Z$. If $m = 1$, the maximal complement is just the equivalence relation $Q$ defined by letting all elements of $W$ be equivalent to a representative of the unique equivalence class in $Z$. So we will assume that $m \geq 2$. Let

$$Q = \{(a_n, b_{n+1}) : 1 \leq n \leq m - 1\} \cup \{(a_m, b_1)\} \cup \{(a_1, y), (y, b_2) : y \in W\} \cup \Delta$$

It is routine to check that $Q$ is indeed a maximal complement for $P$ (just draw a diagram similar to the one at the beginning of this section).

2b. Suppose $Z$ has infinitely many equivalence classes and let $\{a_n, b_n\}$ for $n \in Z$ be a selection of two elements of each equivalence class in $Z$. Then a maximal complement $Q$ for $P$ is defined as follows:

$$Q = \{(a_n, b_{n+1}), (b_{n+1}, a_n) : n \in Z\} \cup \{(a_1, y), (y, a_1), (y, b_2), (b_2, y) : y \in W\} \cup \Delta$$

We show next how the previous result can be extended to separable partial orders.

**Theorem 2.10.** Let $P$ be a pre-order over an infinite set $Y$ such that both $P$ and $P^{-1}$ are separable. Then $P$ has a maximal complement in $PO(Y)$.

**Proof.** Let $D \subseteq Y$ be a countable set which is dense for both $P$ and $P^{-1}$. Let $P_D$ be the restriction of $P$ to $D$. By theorem 2.9 $P_D$ admits a maximal complement $Q^*$ in $PO(D)$. We claim that $Q = Q^* \cup \Delta$ is a maximal complement for $P$ in $PO(Y)$, where $\Delta$ is the identity relation on $Y$. To
see that \( Q \) is a complement of \( P \), it clearly suffices to show that the transitive closure of \( P \cup Q \) is \( Y^2 \). Let \( x, y \in Y \). By the density of \( D \), there are \( d, e \in D \) such that \( (x, d), (e, y) \in P \). Let \( \alpha \) be any path in \( P_D \cup Q \) from \( d \) to \( e \). Then \( \alpha \cup \{(x, d), (e, y)\} \) is the required path from \( x \) to \( y \). To see that \( Q \) is maximal in \( PO(Y) \), it suffices to observe that if \( R \subseteq Q \) is a complement for \( P \) in \( PO(Y) \), then \( R^* = R \cap D^2 \) is a complement for \( P_D \) in \( PO(D) \). In fact, let \( x, y \in D \) and \( \alpha \) be an alternating path in \( R \cup P \) from \( x \) to \( y \). Since \( R \subseteq D^2 \cup \Delta \), then \( \alpha \) is necessarily inside \( D^2 \) and therefore is a path in \( R^* \cup P_D \).

As we have explained in the introduction, the following theorem is a consequence of theorem 2.9

**Theorem 2.11.** Any AT topology over a countable set admits a maximal complement in the lattice of topologies.

*Proof:* Let \((X, \tau)\) be an AT topology and let \( \preceq \) be the associated pre-order (namely, \( x \preceq y \) iff \( x \in \overline{\{y\}} \)) and denote by \( P \) the pre-order \((X, \preceq)\). Let \( Q \) be a maximal complement for \( P \) in \( PO(X) \) given by 2.9. Then the associated AT topology \( \rho \) of \( Q \) is a complement for \( \tau \). To see that \( \rho \) is maximal in \( TOP(X) \) just observe that if \( \rho \subseteq \eta \subseteq \eta' \) and \( \eta' \) is a complement of \( \tau \), then \( \eta \) is also a complement of \( \tau \). So if there is \( V \in \eta' \setminus \rho \), then add \( V \) to \( \rho \) to get an AT complement \( \eta \) of \( \tau \) properly extending \( \rho \). But this would contradict the maximality of \( Q \) in \( PO(X) \). \( \square \)

Our last result is that \( \omega_1 \) with its usual order does not admit a maximal complement in the lattice of pre-orders. We will need the following general fact about maximal complements.

**Proposition 2.12.** Let \( P \) be a total pre-order over a (not necessarily countable) set \( X \) and \( Q \) a maximal complement of \( P \). Then any chain in \( Q \) has size at most two. Moreover, for a given \( x \in X \) there is at most one element \( y \in X \) different than \( x \) such that \( (y, x) \in Q \) and, similarly, there is at most one \( z \) different than \( x \) such that \( (x, z) \in Q \).

*Proof:* Suppose, towards a contradiction, that there are three elements \( a, b \) and \( c \) of \( X \) such that \( a \prec_Q b \prec_Q c \). We will show that \((a, b)\) can be removed from \( Q \) and still yield a complement. Consider the following relation

\[
Q^* = Q \setminus \{(x, b) : (x, b) \in Q \& x \neq b\}
\]

It is clear that \( Q^* \) is transitive. To see that \( Q^* \) is a complement of \( P \), it suffices to show that \((x, b)\) belongs to the transitive closure of \( P \cup Q^* \), for every \((x, b) \in Q \). Let \((x, b) \in Q \) with \( x \neq b \). Since \((b, c) \in Q \) and \( c \neq b \), then \((x, c) \in Q^* \). Since \((b, c) \notin P \) and \( P \) is total, then \((c, b) \in P \) and we are done.

Let \( x \in X \) and suppose there are \( y \) and \( w \) different than \( x \) such that \((y, w) \in P \). We claim that \( Q^* = Q \setminus \{(x, w)\} \) is still a complement of \( P \), which is a contradiction. In fact, it suffices to observe that \((x, w)\) is in the transitive closure of \( P \cup Q^* \). Analogously, suppose that \((x, z), (x, w) \in Q \) for some \( z, w \in X \) different than \( x \). As before, we can assume that \((z, w) \in P \). It is easy to check that \( Q^* = Q \setminus \{(x, w)\} \) is still a complement of \( P \), which is a contradiction. \( \square \)

Let us suppose, towards a contradiction, that \( Q \) is a maximal complement of \( \omega_1 \). Let \( P \) denote the usual order \( < \) of \( \omega_1 \). Since \( 0 \) is the minimal element of \( P \) then there is \( a \in X \) such that \((a, 0) \in Q \). By 2.12 such element \( a \) is unique. For every \( x > a \), let \( C_x = \langle x, x_1, x_2, \ldots, x_n, 0 \rangle \) be a path in \( Q \cup P \) from \( x \) to \( 0 \) of minimal length. By the uniqueness of \( a \) it is clear that \( x_n = a \). Moreover, by 2.12 every chain in \( Q \) has length at most 2, therefore \( n \geq 2 \). We consider two cases:
(i) The set $S$ of all $x \in S$ such that $x_1 < x$ (i.e. the path $C_x$ start in $Q$) is stationary. Let $f(x) = x_1$ be defined in $S$. Since $f$ is regressive by Fodor’s theorem, $f$ takes a constant value $b$ in a stationary set. In particular, there are $b < x < y$ such that $(x, b), (y, b) \in Q$, which contradicts 2.12.

(ii) The set $S$ of all $x > a$ such that $x < x_1$ (i.e. the path $C_x$ start in $P$) is stationary. Consider the function $f(x) = x_2$ for $x \in S$. It is clear by the minimality of $C_x$ that $x_2 < x$, thus $f$ is a regressive function. By Fodor’s theorem, let $T$ be a stationary set where $f$ takes constant value equal to $b$. In this case pick $x, y \in T$ such that $x_1 < y$. Then by the definition of $f$ we have that $(x_1, b), (y_1, b) \in Q$, which contradicts 2.12.

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References

