Mediation in the framework of Morpho-logic

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Abstract. In this paper we introduce a median operator between two sets of interpretations (worlds) in a finite propositional language. Our definition is based on morphological operations and Hausdorff distance. It provides a result which lies "halfway" between both sets and accounts for the "extension" or "shape" of the sets. We prove several interesting properties of this operator and compare it with fusion operators. This new operator allows performing mediation between two sets of beliefs, preferences, demands, in an original way, showing an interesting behavior that was not possible to achieve using existing operators.

1 INTRODUCTION

Consider the situation of two agents who are settling the differences between the demands (goals, etc.) of the members of the parties they represent. One of the negotiating conditions is that each demand should receive a fair treatment. The final output of the process should be a new set of demands (for instance, for the merging of the two parties).

This problem reminds the problem faced by fusion [3] where a finite collection of belief sets is merged into another belief set. However, a general fusion operator will not take into account each belief of each party, only consensual beliefs are important. For instance, let X and Y represent respectively the demands of the parties. When X and Y have non empty intersection, any fusion operator will let $X \cap Y$ to be the result of the fusion. The reason is that fusion relies mostly on the parts of X and Y which are the "closest" to each other. The purpose of this paper is to introduce a *median operator* $\odot$ such that $X \odot Y$ lies "halfway" between X and Y and partly satisfies the demands of every member of both parties. This will be achieved by means of the Hausdorff distance which takes into account the "extension" or "shape" of X and Y. The idea of the median is coming from works in Mathematical Morphology about interpolations between compact sets in metric spaces [6]. The idea of Morpho-logic has been already useful for dealing with others operators that naturally appear in AI problems [2].

As it happens with fusion operators, where beliefs are given at the semantical level, we assume that the sets of demands X and Y are given as sets of interpretations (worlds) in a finite propositional language. We use a distance function d between worlds to measure the similitude between worlds (for all practical purposes the reader can assume that d is the Hamming distance). The distance d gives a method to represent the closest worlds to a set X. In fact, let $\delta(X)$ be the collection of all worlds w such that $d(w, X) \leq 1$, where $d(w, X) = \min\{d(w, x) : x \in X\}$. In morphological terms, $\delta(X)$ is called a dilation of X by a ball of radius 1. More generally, the dilation of X of any radius r is the set $\delta_r(X)$ of all w such that $d(w, X) \leq r$.

The Hausdorff distance between X and Y is the smallest integer ρ such that $X \subseteq \delta_\rho(Y)$ and $Y \subseteq \delta_\rho(X)$. Then ρ measures the compromise that both X and Y should make in order to satisfy all demands of the other party. Our median operator satisfies that $d(z, X \odot Y) \leq \rho/2$ for all $z \in X \cup Y$. It is in this sense that $X \odot Y$ lies "halfway" between X and Y. This is a consequence of the fact that $\odot$ has a stronger property: if $x \in X$ and $y \in Y$ then any path from x to y of length at most ρ will meet $X \odot Y$ (the details are given later).

We first recall, in Section 2, the definition of some fusion operators. In Section 3 we introduce a median operator constructed via dilation and Hausdorff distance, study some of its properties and compare it with the fusion operator. In Section 4 we introduce another median operator constructed via dilation, erosion and Hausdorff distance; we compare this operator to the previous one and study some of its properties. In Section 5 we present a procedural approach to compute $\odot$.

2 FUSION

In this section we recall some previous definitions of fusion [2]. In order to make the exposition more readable, we assume that Ω, the set of worlds, is finite and we fix the distance d as the Hamming distance between worlds. Nevertheless our results hold for all distances $d : \Omega^2 \to \mathbb{N}$ having the following property (of normality): for all $X \subseteq \Omega$ such that $X \neq \emptyset$ and $X \neq \Omega$, there is a $\omega \in \Omega \setminus X$ such that $d(\omega, X) = 1$.

Let us begin recalling the morphological definition [2] of the Δ^{max} operator [3], denoted here by Δ in order to simplify the notation:

$$X \Delta Y = \delta_n(X) \cap \delta_n(Y) \tag{1}$$

where $n = \min\{k : \delta_k(X) \cap \delta_k(Y) \neq \emptyset\}$.

Since dilations are defined using centered balls of a distance d as structuring elements, they are extensive, *i.e.* $X \subseteq \delta_r(X)$ (this property will be often used in the following). Other useful properties of dilation are iterativity ($\delta_r(\delta_{r'}(X)) = \delta_{r+r'}(X)$) and monotony ($X \subseteq Y \Rightarrow \delta_r(X) \subseteq \delta_r(Y)$).

Let $m = d_{\min}(X, Y)$ be the minimum distance between X and Y, *i.e.* $d_{\min}(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$. Our next definition is a variant of Equation (1):

$$X \Delta' Y = \begin{cases} \delta_{m/2}(X) \cap \delta_{m/2}(Y) & \text{if } m \text{ is even,} \\ (\delta_{\lfloor m/2 \rfloor}(X) \cap \delta_{\lfloor m/2 \rfloor + 1}(Y)) & \\ \cup\, (\delta_{\lfloor m/2 \rfloor + 1}(X) \cap \delta_{\lfloor m/2 \rfloor}(Y)) & \text{if } m \text{ is odd.} \end{cases} \tag{2}$$

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Note that if m is even, Δ' and Δ are identical operators. In case m is odd, this new definition avoids dilating "too much". Then Δ' may be preferred to Δ.

In general, $X\Delta'Y \neq X\Delta Y$ if m is odd (in this case, the value of n in equation (1) is $n = \lfloor m/2 \rfloor + 1$). In general we have:

$$X\Delta'Y \subseteq X\Delta Y \tag{3}$$

(with an equality if m is even). This is illustrated in Figure 1.

Note that $X\Delta Y = X \cap Y$ if $X \cap Y \neq \emptyset$ (in this case $n = 0$).

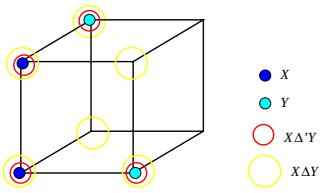

Figure 1. Fusion using Δ and Δ'. The vertices of the cube represent worlds and edges link worlds at distance 1 from each other.

There are some relationships between the two well known fusion operators Δ^{max} and Δ^{Σ} [3]. We have, in fact, the following proposition:

Proposition 1 *For any X and Y, $X\Delta'Y = \Delta^{max}(X,Y) \cap \Delta^{\Sigma}(X,Y)$.*

It has been proved in [4] that Δ^{Σ} is a merging operator and Δ^{max} is a quasi-merging operator. Thus Δ' satisfies all the common postulates (of merging and quasi-merging) that pass to the intersection. For instance, the consistence axiom: $X \cap Y \neq \emptyset$ entails $X\Delta'Y = X \cap Y$.

Next, we introduce some definitions which will be useful in the following. Namely, they will be helpful to show that the operator Δ' can be characterized as in Proposition 2.

A path α between two elements x and y of Ω is a sequence, $x_0, \ldots, x_n$ of points in Ω such that $d(x_i, x_{i+1}) = 1$ for any $i = 0, \ldots, n-1$. The length of such a path α, denoted $|\alpha|$, is n. The set of paths between x and y is denoted $P(x,y)$. If $\alpha = x_0, \ldots, x_n$ is a path, we define the set of central points of α, denoted $c(\alpha)$, as $\{x_{n/2}\}$ if n is even and $\{x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}\}$ otherwise. We will say that an operator Δ_1 is *smaller or equal* than an operator Δ_2 if and only if for any input (X,Y), $X\Delta_1 Y \subseteq X\Delta_2 Y$.

Proposition 2 *The operator Δ' is the smallest operator $\diamond$ having the following property: for all $x \in X$, for all $y \in Y$ and for all path $\alpha \in P(x,y)$, if $|\alpha| \leq m$ then $c(\alpha) \subseteq X \diamond Y$, where $m = d_{\min}(X,Y)$.*

Even if this operator enjoys good properties, it is not adequate for some purposes, namely in order to get a global consensus as the following simple example shows:

Example 3 *Suppose X and Y represent the political agenda of the Labor Party and the Green Party, respectively. They would like to find an agreement in order to present a common platform for the next elections. As it usually happens, there are several leanings in each of the two parties. In order to simplify, suppose there are two leanings in each party, the left and central. Their positions about ten parameters are encoded by worlds. The parameters are the following: nationalization of big private companies; increasing the taxes for the rich people; reducing the taxes for the poor people; increasing the budget of public health; increasing the budget for public education; creating a subvention for the unemployed persons; increasing the budget of research programs; reducing the subsidies for the big agriculture exploitations; creating subsidies for organic agriculture; fostering the public transports. The encoding is made in the following manner: $X = \{w_1, w_2\}$, $Y = \{w_1, w_3\}$, where $w_1 = (1,1,1,1,1,1,1,1,1,1)$ is the encoding of the left wing in both parties, the central wing in the Labor Party is encoded by $w_2 = (0,0,0,1,1,1,1,0,0,0)$ and the central wing in the Green Party by $w_3 = (0,0,0,0,0,0,0,1,1,1)$. If the policy adopted for merging these leanings is a classical fusion, in which the result is $\{w_1\}$ (the intersection of both groups), they run the risk of the splitting of both political parties. Thus they have to adopt a different policy in order to reach an agreement more fair to all leanings.*

The rest of the paper is dedicated to study some operators proposed as an alternative to classical merging operators. These operators are called median operators and take inspiration from Mathematical Morphology [5], in particular [6].

3 MEDIAN FROM HAUSDORFF DISTANCE

One of the main features of the morphological setting is that it offers different ways of talking about closeness. One of them has already been evoked in the previous section when we defined the minimum distance m between X and Y. Actually this notion does not define a distance[4] between elements of $\mathcal{P}^*(\Omega)$, the set of nonempty subsets of Ω (because $m = 0$ iff $X \cap Y \neq \emptyset$). However, there is a true distance between elements of $\mathcal{P}^*(\Omega)$. This distance, d_H, is the so called Hausdorff distance, defined from d in the following way:

$$d_H(X,Y) = \max\{\max\{d(x,Y) : x \in X\}, \max\{d(y,X) : y \in Y\}\} \tag{4}$$

where $d(w,Z) = \min\{d(w,z) : z \in Z\}$, for $w \in \Omega$, $Z \in \mathcal{P}^*(\Omega)$.

It is easy to check that d_H is indeed a distance. In order to simplify the notation, when X and Y are fixed, we put $\rho = d_H(X,Y)$.

Note that, by definition, ρ can be viewed as a measure of the degree of global disagreement between X and Y: among the distances between an element of X and Y and an element of Y and X, ρ is the largest one. Another way to understand the meaning of ρ is given by the following identity:

$$\rho = \min\{n : X \subseteq \delta_n(Y) \wedge Y \subseteq \delta_n(X)\} \tag{5}$$

Thus, the set $\delta_\rho(X) \cup \delta_\rho(Y)$ has a natural "fairness property", but it is in general too big. We propose a better solution guided by the notion of interpolation.

The notion of interpolating set in our discrete case is:

$$Z_n = \delta_n(X) \cap \delta_{\rho-n}(Y) \tag{6}$$

for $n = 0, 1, \ldots, \rho$. When n is roughly $\rho/2$, then the interpolating set Z_n "lies half way between" X and Y. Thus, the definition of our median operator is as follows:

$$X \odot Y = \begin{cases} \delta_{\rho/2}(X) \cap \delta_{\rho/2}(Y) & \text{if } \rho \text{ is even,} \\ (\delta_{\lfloor \rho/2 \rfloor}(X) \cap \delta_{\lfloor \rho/2 \rfloor + 1}(Y)) & \\ \cup\, (\delta_{\lfloor \rho/2 \rfloor + 1}(X) \cap \delta_{\lfloor \rho/2 \rfloor}(Y)) & \text{if } \rho \text{ is odd.} \end{cases} \tag{7}$$

Note that $\lfloor \rho/2 \rfloor + 1 = \rho - \lfloor \rho/2 \rfloor$.

We will show next that $\odot$ is indeed a well defined operator and then show some of its properties.

[4] Remember that a distance have to satisfy $d(X,Y) = 0 \Leftrightarrow X = Y$; $d(X,Y) = d(Y,X)$ and triangle inequality.

Properties

1. Let $Z_{\rho'} = \delta_{\rho'}(X) \cap \delta_{\rho-\rho'}(Y)$ for $\rho' \in \{0, \ldots, \rho\}$. Then $Z_{\rho'} \neq \emptyset$.
   Proof: since the Hausdorff distance is ρ, there exists x in X and y in Y and a path $x_0 = x, ..., x_\rho = y$ of length ρ such that $d(x_i, x_{i+1}) = 1$. For $i = \rho'$, we have $x_{\rho'} \in \delta_{\rho'}(X)$. Since $d(x_{\rho'}, y) \leq \rho - \rho'$, we have also $x_{\rho'} \in \delta_{\rho-\rho'}(Y)$. (The path can be decomposed into a path $x = x_0 ... x_{\rho'}$ of length ρ' and a path $x_{\rho'} ... x_\rho = y$ of length $\rho - \rho'$). Hence $x_{\rho'} \in Z_{\rho'}$. ∎
   This results proves that the definition of $X \odot Y$ is consistent:

$$X \odot Y \neq \emptyset \tag{8}$$

2. Since the dilations are extensive, we have:

$$X \cap Y \subseteq X \odot Y \tag{9}$$

3. Link between median and fusion:
   - if $m < \rho$, then $X \Delta Y \subseteq X \odot Y$ (see Figure 2);
   - if $m = \rho$ and is an even number, then $X \Delta Y = X \odot Y$ (see Figure 3);
   - if $m = \rho = 1$, then $X \odot Y = X \cup Y \subseteq X \Delta Y$.
   - if $m = \rho$ and is an odd number, then $X \odot Y = X \Delta' Y$ and therefore (by Proposition 1) $X \odot Y \subseteq X \triangle Y$.

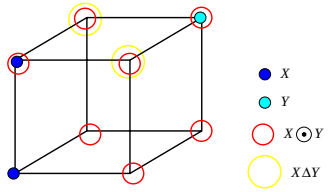

Figure 2. An example where $\rho = 3$ and $m = 2$ and $X \Delta Y \subset X \odot Y$.

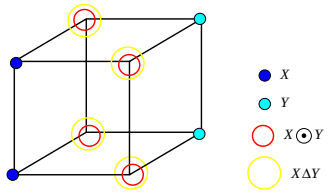

Figure 3. An example where $\rho = m = 2$ and $X \Delta Y = X \odot Y$.

Proof:

- $m < \rho$, ρ and m even: then $n = m/2$, and $X \Delta Y = \delta_{m/2}(X) \cap \delta_{m/2}(Y)$, which is included in $\delta_{\rho/2}(X) \cap \delta_{\rho/2}(Y)$ since $m < \rho$ and dilation is increasing with respect to the size of the structuring element.
- $m < \rho$, ρ odd and m even: then $m \leq \rho - 1$ hence $2n \leq \rho - 1$ and $n \leq \lfloor \rho/2 \rfloor$. Therefore we have $\delta_n(X) \subseteq \delta_{\lfloor \rho/2 \rfloor}(X) \subseteq \delta_{\lfloor \rho/2 \rfloor + 1}(X)$ and the same for Y, from which we deduce $\delta_n(X) \cap \delta_n(Y) \subseteq X \odot Y$.
- $m < \rho$, m odd: then $n = \lfloor m/2 \rfloor + 1$ ($2n = m + 1$), hence $m + 1 \leq \rho$ and $2n \leq \rho$. If ρ is even, then $n \leq \rho/2$ and $X \Delta Y \subseteq X \odot Y$ (as for the first case). If ρ is odd, since $2n$ is even, we have $2n \leq \rho - 1$ and $n \leq \lfloor \rho/2 \rfloor$. Therefore $X \Delta Y \subseteq X \odot Y$ (as for the second case).
- $m = \rho$ and even: then $n = m/2 = \rho/2$ and $\delta_n(X) \cap \delta_n(Y) = \delta_{\rho/2}(X) \cap \delta_{\rho/2}(Y)$.
- $m = \rho = 1$: then $n = 1$ and $\lfloor \rho/2 \rfloor = 0$. Therefore we have $X \subseteq \delta_1(Y)$ and $Y \subseteq \delta_1(X)$ (from the definition of Hausdorff distance) and $X \odot Y = X \cup Y$. Moreover, $X \cup Y \subseteq \delta_1(X)$ and $X \cup Y \subseteq \delta_1(Y)$, so $X \cup Y \subseteq \delta_1(X) \cap \delta_1(Y) = X \Delta Y$. ∎

The last case is illustrated in Figures 4 and 5 in two different situations.

A consequence of this result is that if ω is in $X \Delta Y$ and if $d_{\min}(X, Y) = m$, this does not imply that there is x in X and y in Y such that $d(x, \omega) + d(\omega, y) = m$. An example of ω is shown in Figure 5 for which $d(x, \omega) + d(\omega, y) \geq 2$.

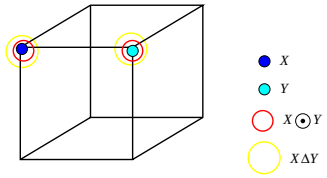

Figure 4. An example where $\rho = m = 1$ and $X \Delta Y = X \odot Y$.

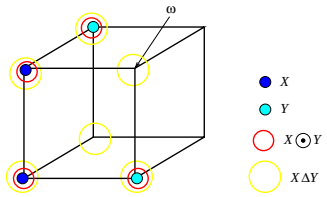

Figure 5. An example where $\rho = m = 1$ and $X \odot Y \subset X \Delta Y$.

Note that this property tells that when $m < \rho$, then $X \Delta Y \subseteq X \odot Y$ and when $m = \rho$, then $X \odot Y \subseteq X \Delta Y$.

4. Commutativity: $X \odot Y = Y \odot X$ (by construction).
5. Since $d_H(X, X) = 0$, we have $X \odot X = X = X \Delta X$.
6. $\odot$ is not associative. A counter-example is shown in Figure 6.

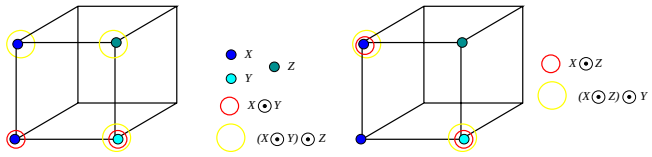

Figure 6. An counter-example of associativity.

7. $\odot$ is not monotone, *i.e.* $X \subseteq X'$ does not entail generally $X \odot Y \subseteq X' \odot Y$.

Example 3 (continued) *In this example we have $\rho = 7$. The point $w = (0, 0, 0, 1, 1, 1, 1, 1, 1, 1) \in X \odot Y$ and it is a distance 3 from w_2 and a distance 4 from w_3. Notice that, unlike the Δ operator, $\odot$ does not forget w_2 and w_3. In fact, for every $z \in X \cup Y$, we can find a point $t \in X \odot Y$ such that $d(z, t) \leq 3$: for $z = w_1$ take $t = w_1$; for $z = w_2$ take $t = w$; for $z = w_3$ take $t = (0, 0, 0, 0, 1, 1, 1, 1, 1, 1)$.*

Now we present the definition of a median operator. It is motivated by the properties of $\odot$. We will show that this notion captures the idea of getting a set that lies halfway between X and Y (see proposition 6).

Definition 4 *An operator $\diamond : \mathcal{P}^*(\Omega) \times \mathcal{P}^*(\Omega) \to \mathcal{P}^*(\Omega)$ is said to be a median operator if $\diamond$ satisfies the following condition:*

$$\forall x \in X, \forall y \in Y, \forall \alpha \in P(x,y) \quad (|\alpha| \leq \rho \Rightarrow c(\alpha) \subseteq X \diamond Y) \tag{10}$$

Theorem 5 *$\odot$ is the smallest median operator,* i.e. *$\odot$ satisfies Equation (10) and for all $X, Y \in \mathcal{P}^*(\Omega)$ and all operator $\diamond$ satisfying Equation (10), $X \odot Y \subseteq X \diamond Y$.*

Proof: It is enough to see that $z \in X \odot Y$ iff $\exists x \in X$, $\exists y \in Y$ and $\exists \alpha \in P(x,y)$ such that $|\alpha| \leq \rho$ and $z \in c(\alpha)$. The *if* part is easy to show: suppose $\alpha = x_0, x_1, \ldots, x_n$ and n is even. Since $z = x_{\frac{n}{2}}$, z is in $\delta_{\frac{n}{2}}(X) \cap \delta_{\frac{n}{2}}(Y)$ and since $n \leq \rho$ and dilations are increasing we have $z \in \delta_{\frac{\rho}{2}}(X) \cap \delta_{\frac{\rho}{2}}(Y)$ (in case ρ is even) or $z \in (\delta_{\lfloor \rho/2 \rfloor}(X) \cap \delta_{\lfloor \rho/2 \rfloor + 1}(Y)) \cup (\delta_{\lfloor \rho/2 \rfloor + 1}(X) \cap \delta_{\lfloor \rho/2 \rfloor}(Y))$ (in case ρ is odd). A similar argument works when n is odd.

For the *only if* part, suppose ρ is even and that $z \in \delta_{\frac{\rho}{2}}(X) \cap \delta_{\frac{\rho}{2}}(Y)$. Thus there are two paths $x_0, x_1, \ldots, x_k = z$ and $z = y_0, y_1, \ldots, y_p$ with $x_0 \in X$, $y_p \in Y$, $k \leq \frac{\rho}{2}$ and $p \leq \frac{\rho}{2}$. So $\alpha = x_0, x_1, \ldots, x_k, y_1, \ldots, y_p$ is a path of length $k + p$. Note that $k + p \leq \rho$. We consider two cases. First, if $|k - p| \leq 1$, $z \in c(\alpha)$. Second, if $|k - p| \geq 2$ the path can be extended, via loops, to a path α' with $|\alpha'| \leq \rho$, $z \in c(\alpha')$ and $\alpha' \in P(x_0, y_p)$ (for instance if $k = p - 2$, $\alpha' = x_0, x_1, x_0, x_1, \ldots, x_k, y_1, \ldots, y_p$). The case ρ odd is similar. ∎

Proposition 6 *Suppose $\diamond$ satisfies Equation (10), then the following holds:*
$d(z, X \diamond Y) \leq \rho/2$, for all $z \in X \cup Y$.

In particular, this is true for $\odot$. Note that the operator Δ' (which satisfies Proposition 2) does not enjoy a property analogous to Proposition 6, namely, there are X and Y and $z \in X \cup Y$ such that $d(z, X \Delta' Y) > m/2$.

4 MEDIAN FROM DILATION AND EROSION

In this section we present a different median operator. We first recall the definition of the erosion $\varepsilon_k(X)$ of size k of a set X: $w \in \varepsilon_k(X)$ if $\delta_k(w) \subseteq X$.

The idea for the new median operator is to use any parameter $\nu \geq m$ in Equation (7) instead of ρ. This method provides a whole family of operators $\odot_\nu$ as follows:

$$X \odot_\nu Y = \begin{cases} \delta_{\nu/2}(X) \cap \delta_{\nu/2}(Y) & \text{if } \nu \text{ is even,} \\ (\delta_{\lfloor \nu/2 \rfloor}(X) \cap \delta_{\lfloor \nu/2 \rfloor + 1}(Y)) & \\ \cup\, (\delta_{\lfloor \nu/2 \rfloor + 1}(X) \cap \delta_{\lfloor \nu/2 \rfloor}(Y)) & \text{if } \nu \text{ is odd.} \end{cases} \tag{11}$$

Note that if $\nu > \rho$, then $\odot_\nu$ is a median operator. For instance, when $\nu = 2\rho$, then $X \cup Y \subseteq \delta_\rho(X) \cap \delta_\rho(Y)$. But the dilations can be very large, and therefore an erosion is applied on the result, leading to the following definition:

$$M'(X,Y) = \varepsilon_k(\delta_\rho(X) \cap \delta_\rho(Y)) \tag{12}$$

This definition is particularly interesting if we take:

$$k = \begin{cases} \rho/2 & \text{if } \rho \text{ is even,} \\ \lfloor \rho/2 \rfloor & \text{if } \rho \text{ is odd.} \end{cases} \tag{13}$$

From the next result and Theorem 2 it follows that M' is indeed a median operator, *i.e.* it satisfies the condition (10).

Theorem 7

$$X \odot Y \subseteq M'(X,Y) \tag{14}$$

Moreover, if $m < \rho$, then $X \Delta Y \subseteq X \odot Y \subseteq M'(X,Y)$.

Proof:

- ρ even, $k = \rho/2$:

  $$X \odot Y \subseteq \varepsilon_{\rho/2}\delta_{\rho/2}(X \odot Y)$$

  since $\varepsilon_{\rho/2}\delta_{\rho/2}$ is a closing, hence extensive. Since $\delta(A \cap B) \subseteq \delta(A) \cap \delta(B)$ for any dilation, we have:

  $$X \odot Y \subseteq \varepsilon_{\rho/2}(\delta_{\rho/2}\delta_{\rho/2}(X) \cap \delta_{\rho/2}\delta_{\rho/2}(Y))$$

  Since $\delta_{\rho/2}\delta_{\rho/2} = \delta_\rho$, we have $X \odot Y \subseteq M'(X,Y)$.
- ρ odd, $k = \lfloor \rho/2 \rfloor$:

  $$X \odot Y \subseteq \varepsilon_{\lfloor \rho/2 \rfloor}\delta_{\lfloor \rho/2 \rfloor}(X \odot Y)$$

  Since δ commutes with $\cup$, we have:

  $$\begin{aligned} \varepsilon_{\lfloor \rho/2 \rfloor}\delta_{\lfloor \rho/2 \rfloor}(X \odot Y) = \varepsilon_{\lfloor \rho/2 \rfloor}(&\delta_{\lfloor \rho/2 \rfloor}(\delta_{\lfloor \rho/2 \rfloor}(X) \cap \delta_{\lfloor \rho/2 \rfloor + 1}(Y)) \\ &\cup\, \delta_{\lfloor \rho/2 \rfloor}(\delta_{\lfloor \rho/2 \rfloor + 1}(X), \delta_{\lfloor \rho/2 \rfloor}(Y))) \end{aligned}$$

  Therefore (with $\lfloor \rho/2 \rfloor + \lfloor \rho/2 \rfloor = \rho - 1$):

  $$\begin{aligned} X \odot Y &\subseteq \varepsilon_{\lfloor \rho/2 \rfloor}((\delta_{\rho-1}(X) \cap \delta_\rho(Y)) \cup (\delta_\rho(X) \cap \delta_{\rho-1}(Y))) \\ &\subseteq \varepsilon_{\lfloor \rho/2 \rfloor}(\delta_\rho(X) \cap \delta_\rho(Y)) \end{aligned}$$

  Indeed:

  $$\begin{aligned} &(\delta_{\rho-1}(X) \cap \delta_\rho(Y)) \cup (\delta_\rho(X) \cap \delta_{\rho-1}(Y)) = \\ &\quad (\delta_{\rho-1}(X) \cup \delta_\rho(X)) \cap (\delta_{\rho-1}(Y) \cup \delta_\rho(Y)) \cap \ldots \\ &\quad = \delta_\rho(X) \cap \delta_\rho(Y) \cap \ldots \end{aligned}$$

  which is included in $\delta_\rho(X) \cap \delta_\rho(Y)$. ∎

Several examples are shown in Figure 7.

5 ORDERINGS AND PROCEDURAL APPROACH

A procedural approach can be derived from the following table ($\Sigma = d(\omega, X) + d(\omega, Y)$ and $Max = \max\{d(\omega, X), d(\omega, Y)\}$):

	X	Y	Σ	Max
ω_1	$d(\omega_1, X)$	$d(\omega_1, Y)$	Σ_1	Max_1
ω_2	$d(\omega_2, X)$	$d(\omega_2, Y)$	Σ_2	Max_2
...				
ω_n	$d(\omega_n, X)$	$d(\omega_n, Y)$	Σ_n	Max_n

The procedures for different operators are as follows:

- Δ-fusion (Δ^{max}): take ω_i having the smallest Max_i. This corresponds to the following total pre-order: $(d_1, d_2) \leq (d'_1, d'_2) \Leftrightarrow \max\{d_1, d_2\} \leq \max\{d'_1, d'_2\}$.
- Σ-fusion (Δ^Σ): take ω_i having the smallest Σ_i. This corresponds to the following total pre-order: $(d_1, d_2) \leq (d'_1, d'_2) \Leftrightarrow d_1 + d_2 \leq d'_1 + d'_2$.
- $\Delta' \subseteq \Delta^\Sigma \cap \Delta^{max}$ corresponds to the following order: $(d_1, d_2) \leq (d'_1, d'_2) \Leftrightarrow d_1 \leq d'_1$ and $d_2 \leq d'_2$. Δ' can be obtained from Σ-fusion and a second filter: $|d(\omega, X) - d(\omega, Y)| \leq 1$ (or minimizing the Max_i).

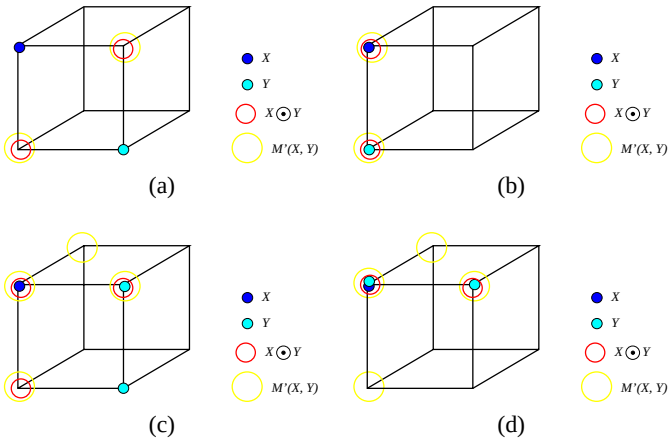

Figure 7. (a) An example with $\rho = m = 2$ and $M'(X,Y) = X \odot Y = X \Delta Y$. (b) An example with $\rho = m = 1$ ($k = 0$) and $M'(X,Y) = X \odot Y = X \Delta Y = X \cup Y$. (c) An example with $\rho = 2$, $m = 1$ ($k = 1$) and $X \odot Y = X \Delta Y$, $X \odot Y \subset M'(X,Y)$. (d) An example with $\rho = 1$, $m = 0$ ($k = 0$), $X \Delta Y = X$, $X \odot Y = X \cup Y$, and $X \Delta Y \subset X \odot Y \subset M'(X,Y)$.

- $\odot$ can be obtained from the table in the following way: first take the set of ω such that $\Sigma \leq \rho$ then among these worlds take those satisfying $|d(\omega, X) - d(\omega, Y)| \leq 1$.
- Note that ρ can be calculated using the table: ρ is the maximum in the column of Max for the rows corresponding to worlds of $X \cup Y$.

One of the main features of this procedural approach is that it suggests very precise manners to extend the median operators to more than two elements.

Extensions to more than two elements $X_1 ... X_k$: We propose three possible extensions:

1. Based on Proposition 1, we extend Δ' using the following procedure: minimize the column of Σ and then minimize the Max.
2. ρ can be generalize as follows:

$$\rho = \max_i \{d(\omega, X_i) : \omega \in X_1 \cup ... \cup X_k\} \qquad (15)$$

This number corresponds to the maximum of the columns of Max among the rows coming from worlds in $X_1 \cup ... \cup X_k$.
Let $\bar{\rho} = (k-1)\rho$. First, take all ω such that $\Sigma \leq \bar{\rho}$. For the second step we have two alternatives: (i) keep only those ω such that the sum of $|d(\omega, X_i) - d(\omega, X_j)|$ is minimal; (ii) keep only those ω such that $|d(\omega, X_i) - d(\omega, X_j)| \leq k - 1$.
Examples are shown in Figures 8 and 9.

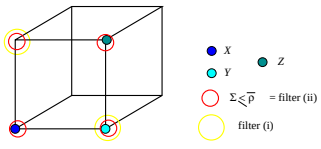

Figure 8. An example of extension to three elements.

3. First compute:

$$A_{ij} = X_i \odot_{2\rho} X_j = \delta_\rho(X_i) \cap \delta_\rho(X_j) \qquad (16)$$

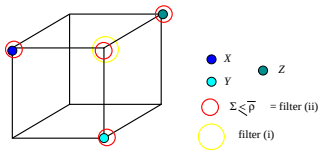

Figure 9. Another example of extension to three elements.

where ρ is given by Equation 15. The proposed operator is $\bigcap_{i \neq j} A_{ij}$. It is easy to show that this intersection is non empty. Notice also that if we take $X_i \odot_\rho X_j$ in (16) the intersection could be empty.
Since the intersection could be very large, we can then take the most central part (the last non empty erosion). This is similar to the idea of barycenter. An example is shown in Figure 10.

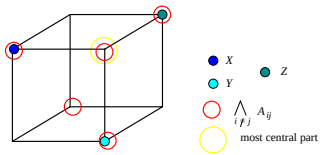

Figure 10. Extension to three elements using the last approach.

6 CONCLUSION

We have proposed the notion of median operator. The minimal of all median operators is denoted by $\odot$. The operator $\odot$ is useful for computing the "middle interpolators" between two sets of beliefs, preferences or demands. This provides an original way to perform mediation or negotiation, showing interesting results which were not possible to achieve using existing operators. In particular we have shown that all parts of the two input sets are taken into account in the final decision, as opposed to classical fusion operators which account only for the parts that are the closest to each other.

Future work aims at further exploring the proposed extensions to more than two sets. Other definitions could also be developed, based on different morphological approaches, such as influence zones [1, 7]. Further applications to negotiations problems are also foreseen.

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