Smooth Sets for Borel Equivalence Relation

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Abstract

We study some properties of smooth Borel set with respect to a Borel equivalence relation, showing some analogies with the collection of countable sets from a descriptive set theoretic point of view. We found what can be seen as an analog of the hyperaritmectic reals in the context of smooth sets. We also present some results about the σ-ideal of closed smooth sets.

1 Introduction

The study of Borel equivalence relation has received recently a lot of attention (see [3] and the references given there). Our basic reference for equivalence relations is [9] and concerning the descriptive set theory we follow the notation of [11]. Let X be a Polish space . A Borel equivalence relation E (Borel as a subset of $X \times X$) on a Borel set B of X is said to be *smooth* if it admits a countable Borel separating family, i.e., a collection (A_n) of E-invariant Borel subsets of B such that for all $x, y \in B$

$$xEy \text{ if and only if } (\forall n)(x \in A_n \leftrightarrow y \in A_n).$$

Given an arbitrary Borel equivalence relation E on X, a set $A \subseteq X$ is called *E-smooth* if there is a Borel set $B \supseteq A$ such that the restriction of E to B is a smooth equivalence relation. The collection of E-smooth sets forms a σ-ideal. Thus we consider smoothness a notion of smallness. Smooth sets are a generalization of the notion of wandering sets in ergodic theory (see [12]).

A basic result about smooth equivalence relations is the Glimm-Effros type Dichotomy Theorem proved by Harrington, Kechris and Louveau in [6], which characterizes the smooth Borel equivalence relations and thus the Borel smooth sets. This theorem can be extended to $\mathbf{\Sigma^1_1}$ sets (see theorem 2.4), this result can be considered as an analog of the Perfect Set Theorem in the context of smoothness. Also we present what could be thought as an analog of the hyperarithmetic reals (see 2.6(iii) and 3.7).

Theorem 2.4 will also provide the basic representation of $\mathbf{\Sigma^1_1}$ smooth sets as the common null sets for the family of E-ergodic non-atomic measures. In particular, it says that smoothness for $\mathbf{\Sigma^1_1}$ sets is a notion concentrated on closed sets, i.e., a $\mathbf{\Sigma^1_1}$ set A is smooth if and only if every closed subset of A is smooth. We called the sets with this property *sparse* sets and they are the analog of thin sets (i.e., sets without perfect subset). Smoothness and Sparseness are not equivalent for Π^1_1 sets in general in ZFC (see [13]), a similar result as for thin sets. Becker (see [1]) has proved recently that these notions are not equivalent even assuming the axiom of Determinacy (contrary to what happens with thin sets). In §3 we will look at the particular case of a countable equivalence relation (i.e., one all of whose equivalence class are countable), presenting a characterization of Borel smooth sets in terms of a notion that generalizes the concept of recurrent point (see [12]). Since smoothness for analytic sets is concentrated on closed sets we will look in §4 at the σ-ideal of closed smooth sets . Following ideas from [10] and [14] we will show that it is a strongly calibrated, locally non-Borel, Π^1_1 σ-ideal. The results presented are part of my Ph.D thesis, I would like to thank my adviser Dr. Alexander Kechris for his guidance and patience.

2 Smooth sets

First we will define the basic concepts and state some basic facts. Let X be a Polish space (i.e., a complete separable metric space). E will always denote an Borel equivalence relation on X. $[x]_E$ or sometimes E_x will denote the E-equivalence class of x. $[A]_E$ is the saturation of A, i.e., $[A]_E = \{y \in X : \exists x \in A(xEy)\}$. A set A is called E-invariant (or just invariant, if there is no confusion about E), if $A = [A]_E$. Given a Δ^1_1 equivalence relation E, (i.e., E as a subset of $X \times X$ is a Δ^1_1 set) and $A \subseteq B$, with B a Π^1_1 invariant set and A a Σ^1_1 set, then there is a Δ^1_1 invariant set C with $A \subseteq C \subseteq B$. In other words, the separation theorem holds in an invariant form for Δ^1_1 equivalence relations (actually it holds for Σ^1_1 equivalence relations), a proof of this can be found in [6] (lemma 5.1). We will use the following notation: Script capital letters will denote a countable family of subsets of X, i.e., $\mathcal{A} = (A_n)$, with $A_n \subseteq X$ for $n \in \mathbf{N}$. For each of these collections we define the following equivalence relation:

$$x \, E_{\mathcal{A}} \, y \quad \text{if and only if} \quad (\forall n)(x \in A_n \longleftrightarrow y \in A_n).$$

Definition 2.1 *Let Γ be a pointclass*
(i) E is Γ-separated *if and only if there is a countable collection $\mathcal{A} = (A_n)$ with each $A_n \in \Gamma$, such that: $\forall x \forall y (xEy \longleftrightarrow x \, E_{\mathcal{A}} \, y)$, i.e., $E = E_{\mathcal{A}}$.*
(ii) A subset A of X is Γ-separated, *if and only if there is a collection $\mathcal{A} = (A_n)$ of E-invariant sets, with each $A_n \in \Gamma$, and $\forall x \in A, \forall y \in A (xEy \longleftrightarrow x \, E_{\mathcal{A}} \, y)$. In this case we say that $\mathcal{A}$* separates A.
(iii) A is called strongly Γ-separated *if $\forall x \in A \, \forall y (xEy \longleftrightarrow x \, E_{\mathcal{A}} \, y)$; and we say that $\mathcal{A}$* strongly separates A.

Remarks: (1) Notice that in (i), each A_n has to be E-invariant (because if $x \in A_n$ and yEx, then $x \, E_{\mathcal{A}} \, y$. Hence $y \in A_n$).

(2) Denote by $[x]_{\mathcal{A}}$ the $E_{\mathcal{A}}$-equivalence class of x. Then $\mathcal{A}$ separates A if and only if for all $x \in A$, $[x]_E \cap A = [x]_{\mathcal{A}} \cap A$; and $\mathcal{A}$ strongly separates A if and only if for all $x \in A$, $[x]_E = [x]_{\mathcal{A}}$.

(3) If $\mathcal{A} = (A_n)$ and each A_n is invariant then $E \subseteq E_{\mathcal{A}}$, thus only one direction in (ii) is not trivial.

A finite, positive Borel measure μ on X is called E-*ergodic* if for every μ-measurable invariant set A, $\mu(A) = 0$ or $\mu(X - A) = 0$. It is called E-*non atomic*, or just *non atomic*, if for every $x \in X$ $\mu([x]_E) = 0$. A basic fact about E-ergodic non-atomic measure is that if μ is such a measure, then there is no μ-measurable separating family for E. A typical example of an equivalence relation with a non atomic ergodic measure is E_0, which is defined on 2^ω by

$$xE_0 \, y \quad \text{if and only if} \quad \exists m \, \forall n > m \; (x(n) = y(n)).$$

The usual product measure on 2^ω is non atomic and E_0-ergodic (the so called 0-1 law).

One way of defining ergodic measures is through embeddings. Let E and $E^{'}$ be two equivalence relations on X and Y respectively. An *embedding* from E into $E^{'}$ is a 1-1 map $f : X \to Y$ such that for all $x, y \in X$, $xEy \longleftrightarrow f(x)E^{'}f(y)$. For Borel equivalence relations we define $E \sqsubseteq E^{'}$ if there is a Borel embedding of E into $E^{'}$.

A fundamental result about these notions is the following theorem of Harrington, Kechris and Louveau (see [6]). We will refer to it as the HKL theorem.

Theorem 2.2 *(Harrington,Kechris,Louveau [6]) Let X be a recursively presented perfect Polish space, E a Δ^1_1 equivalence relation on X. Then exactly one of the following holds:*

(1) *E has a Δ^1_1 separating family $\mathcal{A} = (A_n)$, such that the relation "$x \in A_n$", is Δ^1_1.*

(2) *$E_0 \sqsubseteq E$ (via a continuous embedding).*

□

We are interested on the restriction of the Borel equivalence relation to a subset of X

Definition 2.3 *Let Γ be a pointclass*
(i) Let $A \subseteq X$ and define $E\lceil A$ to be the restriction of E to A, i.e., $E\lceil A = E \cap (A \times A)$. $E\lceil A$ is an equivalence relation on A. And, naturally, we say $E\lceil A$ is Γ-separated if there is a countable collection $\mathcal{A} = (A_n)$ of Γ-subsets of A such that for all $x, y \in A$ ($xE_{\mathcal{A}}\, y \longleftrightarrow xEy$).
(ii) A measure μ on X is called $E\lceil A - ergodic$ if $\mu(X - A) = 0$ and for every $B \subseteq A$ which is $E\lceil A$-invariant and μ-measurable, we have $\mu(B) = 0$ or $\mu(X - B) = 0$. Notice that $\mu(X - B) = 0$ if and only if $\mu(A - B) = 0$.

If $A \in \Gamma$ (for Γ a pointclass closed under intersections) is invariant, then it is clear that A is Γ-separated if and only if $E\lceil A$ is Γ-separated. The next theorem says, among other things, that for a Borel equivalence relation all the natural variations for a notion of countable separation for $\boldsymbol{\Sigma}^1_1$ sets are equivalent.

Theorem 2.4 *Let X be a recursively presented Polish space, E a Δ^1_1 equivalence relation on X, and A a Σ^1_1 subset of X. The following are equivalent:*

(1) *There is a Δ^1_1 invariant set B such that $A \subseteq B$ and B is (strongly) Δ^1_1-separated. Moreover, the separating family for B is uniformly Δ^1_1, i.e., the relation (in x and n) "$x \in A_n$" is Δ^1_1 .*

(2) *A is strongly Δ^1_1-separated.*

(3) *$[A]_E$ is Σ^1_1-separated.*

(4) *A is Σ^1_1-separated.*

(5) *$E\lceil A$ is Σ^1_1-separated.*

(6) *A is universally measurable separated.*

(7) *$E\lceil A$ is universally measurable separated.*

(8) *For every E-ergodic non atomic measure μ, $\mu(A) = 0$.*

(9) *For every $E\lceil A$-ergodic, non atomic measure μ, $\mu(A) = 0$.*

(10) *$E_0 \not\sqsubseteq E\lceil A$.*

Similarly, the same equivalence holds by relativization for a $\boldsymbol{\Sigma^1_1}$ set A and a $\boldsymbol{\Delta^1_1}$ equivalence relation.

Proof: All the equivalence are more or less straightforward, except for (10) $\Rightarrow$ (1) which uses two results coming from the proof of the HKL theorem 2.2. That proof uses the Gandy-Harrington topology (also called the Σ^1_1-topology). The basis for this topology is the collection of Σ^1_1 sets. This is a Baire topology (i.e., it satisfies the Baire category theorem), the basic facts about it can be found in [6].

Lemma A: *Let τ be the Gandy-Harrington topology on X and $\overline{E}$ the $\tau \times \tau$-clousure of E. Let A be a Σ^1_1 subset of X. If $\{x : E_x \neq (\overline{E})_x\} \cap A \neq \emptyset$ then $E_0 \sqsubseteq E\lceil A$, via a continuous embedding.*

Proof: In the proof of the HKL theorem was shown that if $\{x : E_x \neq (\overline{E})_x\} \cap A \neq \emptyset$, then E is meager in $(A \times A) \cap \overline{E}$ (see lemma 5.3 of [6]). Hence the construction of the embedding from E_0 into $E\lceil A$ can be carried out in A.

($\Box$ lemma A)

Lemma B: *Let $D = \{x : E_x = (\overline{E})_x\}$, D is a Π^1_1 strongly Δ^1_1-separated invariant set. Actually, the separating family for D is $\{A \subseteq X : A$ is a Δ^1_1 invariant set $\}$.*

Proof: First, $\overline{E}$ is a Σ^1_1 equivalence relation (see lemma 5.2 of [6]). And we have: $x \in D$ if and only if $\forall y\ (x\overline{E}y \to xEy)$. Thus D is Π^1_1 . Also, as $E \subseteq \overline{E}$, then D is E-invariant (actually $\overline{E}$-invariant). On the other hand, we know $\overline{E} = \sim \bigcup\{A \times \sim A : A$ is Δ^1_1 invariant set $\}$. So, if $\mathcal{A} = \{A : A$ is a Δ^1_1 invariant set $\}$, then $\overline{E} = E_{\mathcal{A}}$. And we get: $\forall x \in D(E_x = (\overline{E})_x = (E_{\mathcal{A}})_x)$. Thus $\forall x \in D\ \forall y(xE_{\mathcal{A}}y \longleftrightarrow xEy)$, i.e., D is strongly separated by $\mathcal{A}$.

($\Box$ lemma B)

Now we finish the proof of (10) $\Rightarrow$ (1). Suppose (10) holds. Then by Lemma A $A \subseteq D$. By separation there is a Δ^1_1 invariant set B with $A \subseteq B \subseteq D$. Hence, by lemma B B is strongly Δ^1_1 separated by $\{A \subseteq X : A$ is Δ^1_1 invariant set $\}$. Now, $\mathcal{A}$ is clearly a Π^1_1 collection, so by a separation argument (see [6]) we can easily show that there is a Δ^1_1 subsequence of $\mathcal{A}$ which also separates B, so (1) holds.

$\Box$

In view of this result we have

Definition 2.5 *(i)Let E be a Borel equivalence relation on X. A $\boldsymbol{\Sigma^1_1}$ subset $A \subseteq X$ is called* E-smooth *if any of the equivalent conditions of theorem 2.4 holds.*
(ii) A set $A \subseteq X$ is called E-smooth *if there is a Borel smooth set B such that $A \subseteq B$*

It is clear that a subset of a smooth set is also smooth and countable unions of smooth sets are smooth. So, we regard smooth sets as small sets. There is a strong similarity between the collection of countable sets and the collection of $\boldsymbol{\Sigma^1_1}$ smooth sets, which is summarized in the following

Theorem 2.6 *Let E be a Δ^1_1 equivalence relation on a recursively presented Polish space X.*
(i) (Analog of the perfect set theorem for $\boldsymbol{\Sigma^1_1}$ sets) Let $A \subseteq X$ be a Σ^1_1 set. Then either A is smooth or $E_0 \sqsubseteq E\lceil A$ (via a continuous embedding). Similarly the same result holds by relativization for a $\boldsymbol{\Sigma^1_1}$ set A and a $\boldsymbol{\Delta^1_1}$ equivalence relation E.

(ii) The collection of $\boldsymbol{\Sigma}_1^1$ *smooth sets is* Π_1^1 *on the codes of* $\boldsymbol{\Sigma}_1^1$ *sets.*
(iii) (Analog of the hyperarithmetic reals) Let $\overline{E}$ *be the* $\tau \times \tau$*-closure of* E*, where* τ *is the GH-topology on* X*. Put*

$$D = \{x : E_x = (\overline{E})_x\}$$

then D *is a* Π_1^1 *set and for every* Σ_1^1 *set* A*,* A *is smooth if and only if* $A \subseteq D$*.*
□

Proof: (i) It follows from 2.4.

(ii) Let $\mathcal{U}$ be a Σ_1^1 universal set, then from 2.4 we have that

$$\mathcal{U}_\alpha \text{is smooth if and only if } \exists \mathcal{A} \in \Delta_1^1(\alpha)\, \forall x, y \in \mathcal{U}_\alpha\ (xEy \longleftrightarrow x\, E_{\mathcal{A}}\, y) (*)$$

It is easy to see that $(*)$ is a Π_1^1 relation by coding sequences of $\Delta_1^1(\alpha)$ invariants sets.

(iii) It follows from lemmas A and B in the proof of 2.4

□

The set D is the largest strongly Δ_1^1 separated set. In fact: Let $\mathcal{A} = \{A : A \text{ is } \Delta_1^1 \text{ invariant set }\}$ and B a strongly Δ_1^1-separated set, say by a family $\mathcal{B}$ of Δ_1^1 invariant sets. Let $D_{\mathcal{B}} = \{x : [x]_E = [x]_{\mathcal{B}}\}$, i.e., $x \in D_{\mathcal{B}}$ if and only if for all $y(x\, E_{\mathcal{B}}\, y \longleftrightarrow xEy)$. Analogously we define $D_{\mathcal{A}}$. We saw in 2.4 lemma B that $D = D_{\mathcal{A}}$. By definition of strong separation $B \subseteq D_{\mathcal{B}}$. But as $\mathcal{B} \subseteq \mathcal{A}$, then $E_{\mathcal{A}} \subseteq E_{\mathcal{B}}$ and thus $D_{\mathcal{B}} \subseteq D_{\mathcal{A}}$. Therefore $B \subseteq D_{\mathcal{A}}$.

Let us recall here that the collection of hyperarithmetic reals, denoted by $\Delta_1^1(X)$, is a true Π_1^1 set and is equal to $\bigcup\{A : A \text{ is a countable } \Delta_1^1 \text{ set }\}$. Continuing the analogy with the collection of countable sets we have the following natural questions:

(i) Is $D = \bigcup\{A : A \text{ is } \Delta_1^1 \text{ smooth set}\}$? Equivalently, is D the union of Σ_1^1 sets?

(ii) Is D a true Π_1^1 set ?

We will show in §3 that for a countable Δ_1^1 equivalence relation the answer for (i) is yes. And as a consequence of a theorem of Kechris, this is also true for a Δ_1^1 equivalence relation generated by the action of a locally compact group of Δ_1^1 automorphisms of X. Regarding question (ii), we know that for E_0 D is a true Π_1^1 set, which shows that in this case the analogy between D and the hyperarithmetic reals is quite clear. The proof of this is as follows: Every Δ_1^1 point $x \in 2^\omega$ belongs to D; since $\{x\}$ is a Δ_1^1 smooth set. Also,

D has measure zero with respect to the standard product measure on 2^ω (because this measure is E_0-ergodic). Then by a basis theorem it cannot be Δ^1_1, otherwise its complement would contain a Δ^1_1 point.

3 The case of a countable Borel equivalence relation

In this section we will look at the particular case of a countable Borel equivalence relation, i.e., one for which every equivalence class is countable. Typical examples are equivalence relations generated by a Borel automorphism (i.e., hyperfinite equivalence relations), and more generally by the action of a countable group of Borel automorphisms. In fact, a theorem of Feldman-Moore (see [5]) says that for every countable Borel equivalence relation E on a Polish X there is a countable group G of Borel automorphisms of X such that $E = E_G$, where

$$xE_Gy \text{ if and only if } g(x) = y, \text{ for some } g \in G.$$

It is a classical fact that for every Borel subset B of X there is a Polish topology τ, extending the given topology of X, for which B is τ-clopen. Moreover, τ admits a basis consisting of Borel sets with respect to the original topology of X. Thus the Borel structure of X is not changed. As a corollary we get that for every countable Borel equivalence relation E there is a Polish topology τ and a countable group G of τ-homeomorphisms of X such that $E = E_G$, τ extends the original topology of X and the Borel structure of X remains the same. These results have an effective version. The Feldman-Moore result quoted above has an effective proof. That is to say: If E is a Δ^1_1 countable equivalence relation, then there is a countable group G of Δ^1_1-automorphisms of X such that $E = E_G$. Moreover, there is a Δ^1_1 relation $R(x, y, n)$ on $X \times X \times \omega$ such that for all n, R_n is a graph of some $g \in G$. And vice versa, for all $g \in G$ there is n such that $\mathrm{graph}(g) = R_n$. By an abuse of the language we will say that the relation $R(g, x, y) \Leftrightarrow g(x) = y$ is Δ^1_1.

Notice that in this case if $Q(x)$ is a Δ^1_1 relation, then $\exists g \in G\ Q(g(x))$, $\forall g \in G\ Q(g(x))$ are also Δ^1_1. In other words $\exists y \in [x]_E\ Q(y)$ and $\forall y \in [x]_E\ Q(y)$ are Δ^1_1.

Also if $R(x, y, n)$ is a Δ^1_1 enumeration of G as above, then there is a Polish topology τ extending that on X such that every $g \in G$ is a τ-homeomorphism and τ admits a basis of Δ^1_1 sets effectively enumerated. The classical proofs of this facts can be found in [5] and [12], and for the effective counterpart see [9] and [13]. As a corollary of this results we have

Proposition 3.1 *The collection of Δ^1_1 sets forms a basis for a Polish topology τ such that every Δ^1_1 set is τ-clopen.*

We will state the result we will need without a proof, since it is a consecuence of the effective results mentioned above.

Proposition 3.2 *Let E be a Δ^1_1 countable equivalence relation on X, $B \subseteq X$ a Δ^1_1 set and G a countable group of Δ^1_1 automorphisms of X such that $E = E_G$ with "$g(x) = y$ " a Δ^1_1 relation (as it was explained above). There is a Polish topology τ extending that on X such that every $g \in G$ is a τ-homeomorphism and $[B]_E$ is τ-clopen. Moreover, τ admits a basis of Δ^1_1 sets effectively enumerated.*

The following definitions will play a crucial role in the sequel.

Definition 3.3 *Let τ be a Polish topology on X and put*

$$P(\tau) = \{x \in X : [x]_E \text{ has an isolated point with respect to } \tau \}$$

Notice that in the case of E generated by a single homeomorphism of (X, τ), $X - P(\tau)$ is a generalization of the notion of recurrent points (see [12]). Recall that for each countable collection $\mathcal{A} = (A_n)$ of E-invariant sets we define an equivalence relation $x \, E_{\mathcal{A}} \, y$ by

$$x \, E_{\mathcal{A}} \, y \text{ if and only if } \forall n \, (x \in A_n \longleftrightarrow y \in A_n).$$

and denote the $E_{\mathcal{A}}$-equivalence classes by $[x]_{\mathcal{A}}$.

Definition 3.4 *For each countable collection $\mathcal{A} = (A_n)$ of E-invariant sets put*

$$D_{\mathcal{A}} = \{x \in X : [x]_E = [x]_{\mathcal{A}}\}$$

i.e., $x \in D_{\mathcal{A}}$ if and only if $\forall y \, (xEy \longleftrightarrow x \, E_{\mathcal{A}} \, y)$.

Notice that a set B is strongly separated by $\mathcal{A}$ if and only if $B \subseteq D_{\mathcal{A}}$. The following result will be very important in the sequel.

Proposition 3.5 *Let E be a countable equivalence relation on X, τ a Polish topology on X with basis $\{W_n : n \in \mathbf{N}\}$ such that the E-saturation of every τ-open set is τ-open. Put $B_n = [W_n]_E$ and $\mathcal{B}$ =(B_n). Then $P(\tau) = D_{\mathcal{B}}$.*

Proof: First we prove that if $y \notin D_{\mathcal{B}}$, then $y \notin P(\tau)$. It suffices to show that if $x \notin D_{\mathcal{B}}$ and $x \in W_n$, then $\mid W_n \bigcap [x]_E \mid > 1$. This is because if $y \notin D_{\mathcal{B}}$ and $W_n \bigcap [y]_E \neq \emptyset$, say $x \in W_n \bigcap [y]_E$, then as $D_{\mathcal{B}}$ is invariant $x \notin D_{\mathcal{B}}$, and so $\mid W_n \bigcap [y]_E \mid = \mid W_n \bigcap [x]_E \mid > 1$.

So, suppose $x \notin D_{\mathcal{B}}$ and let y be such that $x \, E_{\mathcal{B}} \, y$ but $x \not\!E y$. Let n be such that $x \in W_n$. So, in particular $W_n \neq \{x\}$: Otherwise $x \in D_{\mathcal{B}}$ (let us observe that (X, τ) can have isolated points). As $y \in [W_n]_E$, there is $w \in W_n$ with yEw. Clearly $x \not\!E w$ and $x \, E_{\mathcal{B}} \, w$. Put $V = [W_n]_E - \{x\}$; V is τ-open and $V \cap W_n \neq \emptyset$. Thus there is m such that $w \in W_m \subseteq V \cap W_n$, but as $x \, E_{\mathcal{B}} \, w$ then $x \in [W_m]_E$. Therefore for some $z \in W_m$ zEx. Clearly $x \neq z$, hence $\mid W_n \bigcap [x]_E \mid > 1$, i.e., $x \notin P(\tau)$.

Second, we show that if $x \in D_{\mathcal{B}}$ then $x \in P(\tau)$. Let $x \in D_{\mathcal{B}}$. Then $[x]_E = [x]_{\mathcal{B}}$ and hence $[x]_E = \{y : (\forall n)(x \in B_n \leftrightarrow y \in B_n)\}$. As each B_n is τ-open, $[x]_E$ is a τ-G_δ set. Since $[x]_E$ is countable, by the Baire category theorem we conclude that $[x]_E$ has a τ-isolated point, i.e., $x \in P(\tau)$.

□

Notice that $P(\tau) \subseteq D_{\mathcal{B}}$ is always true, without assuming that E is countable.

Theorem 3.6 *Let τ be a Polish topology on X with a basis consisting of Borel sets with respect to the original topology on X. Let G be a countable group of τ-homeomorphisms of X and $E = E_G$. Then a τ-G_δ E-invariant set H is E-smooth if and only if $H \subseteq P(\tau)$.*

Proof: Let $\mathcal{B}$ be as in lemma 3.5, then $P(\tau) \subseteq D_{\mathcal{B}}$. As each element of the basis of τ is Borel, we get that $P(\tau)$ is strongly Borel separated.

On the other hand, suppose H is E-smooth, by a result of Effros [4] we get that for every $x \in H$, $[x]_E$ is τ-locally closed in H. But as H is τ-G_δ and $[x]_E$ is countable, then $[x]_E$ has a τ-isolated point, i.e., $x \in P(\tau)$.

□

As a corollary we get the following characterization of Borel smooth sets.

Corollary 3.7 *Let E be a $\mathbf{\Delta^1_1}$ countable equivalence relation on X and B a $\mathbf{\Delta^1_1}$ subset of X. Let τ_B be the canonical Polish topology for $[B]_E$ given by 3.2. Then B is smooth if and only if $B \subseteq P(\tau_B)$.*

Proof: Since $[B]_E$ is τ_B-clopen, by the previous theorem $[B]_E$ is smooth if and only if $[B]_E \subseteq P(\tau_B)$. And by 2.4 B is smooth if and only if $[B]_E$ is smooth. Finally observe that $P(\tau)$ is an invariant set, thus $B \subseteq P(\tau_B)$ if and only if $[B]_E \subseteq P(\tau_B)$.

□

Remark: (i) This corollary can be seen as a Borel analog of 2.6(iii). That is to say for Borel smooth sets $P(\tau)$ plays the same role as D does for Σ^1_1 smooth sets. We will show below that in this case we have that $D = P(\tau)$ for some topology.

(ii) On the other hand this is a generalization of a result of Weiss (see [12]) which says that the equivalence relation induced by an aperiodic homeomorphism is not smooth if and only if there is a recurrent point.

Our next theorem answers a question raised in §2.

Theorem 3.8 *Let E be a countable Δ^1_1 equivalence relation on a recursively presented Polish space X. Let D be the set defined on 2.6(iii) and ρ be the Polish topology generated by the Δ^1_1 sets (see 3.1). Then*

(i) $D = P(\rho)$

(ii) $D = \bigcup\{A : A \text{ is a } \Delta^1_1 \text{ smooth set}\}$

Proof: Let us show first that (i) implies (ii). Let $x \in D$. We want to show that there is a Δ^1_1 smooth set A with $x \in A$. Since $[x]_E$ has a ρ-isolated point, let B be a Δ^1_1 set such that $\mid B \cap [x]_E \mid = 1$. Put $A = \{y :\mid B \cap [y]_E \mid = 1\}$. It is easy to check that A is Δ^1_1: just recall that $\exists z \in [y]_E$ and $\forall z \in [y]_E$ are number quantifiers. Clearly $A \subseteq P(\rho) = D$, so A is smooth and $x \in A$.

Let $\mathcal{A} = (A_n)$ be the collection of Δ^1_1 invariant sets. It follows from the proof of 2.4 Lemma B that $D = D_{\mathcal{A}}$. For every Δ^1_1 set A, $[A]_E$ is Δ^1_1. Hence from 3.5 we get that $D = P(\rho)$.

□

As we have observed before, the previous theorem implies that strong Borel separation and smoothness are equivalent

Theorem 3.9 *Let E be a Δ^1_1 countable equivalence relation on X and C be an arbitrary subset of X. Then C is smooth if and only if C is strongly Borel separated.*

Proof: (i)⇒ (ii) is a consequence of 2.4, as Δ^1_1 smooth sets are clearly Δ^1_1 strongly separated.

(ii)⇒ (i). Let C be a Δ^1_1 strongly separated set. Since D is the largest Δ^1_1 separated set, we have $C \subseteq D$, and from the previous result we have that D is Borel.

□

This result is not valid if we replace strong separation by separation, see [1].

4 The σ-ideal of closed smooth sets

As we have already pointed out, theorem 2.4 implies that the notion of smoothness for $\mathbf{\Sigma^1_1}$ sets is concentrated on closed sets, i.e., a $\mathbf{\Sigma^1_1}$ set A is smooth if and only if every closed subset of A is smooth. In this part we will deal with the collection of closed smooth sets. To be more precise, let E be a Borel equivalence relation on a compact Polish space X. The collection of closed subsets of X, which is denoted by $\mathcal{K}(X)$, equipped with the Hausdorff topology is a Polish space. Let

$$I(E) = \{K \in \mathcal{K}(X) : K \text{ is smooth with respect to } E\}.$$

It is clear that $I(E)$ is a σ-ideal. We are interested in studying the complexity of $I(E)$ as well as some of its structural properties like calibration, the covering property and Borel basis, we will follow the ideas from [10] and [14]. One of the results of this section is that E is smooth if and only if $I(E)$ is Borel. We will also look at the particular case of $I(E_0)$.

A Π^1_1 σ-ideal I satisfies the so called dichotomy theorem (see [10]), namely either I is a true Π^1_1 set or a G_δ set.

Theorem 4.1 *Let E be a non smooth Δ^1_1 equivalence relation on a compact Polish space X. Then $I(E)$ is a strongly calibrated, locally non Borel, Π^1_1 σ-ideal .*

To show that $I(E)$ is locally non Borel we need the following two lemmas.

Lemma 4.2 *Let $f : 2^\omega \to X$ be a continuous embedding from E_0 into E. For every closed set $K \subseteq 2^\omega$*

$$K \in I(E_0) \textit{ if and only if } f[K] \in I(E).$$

Proof: Let $K \notin I(E_0)$ and put $E_1 = E_0\lceil K$. By 2.4, $E_0 \sqsubseteq E_1$ via a continuous embedding. But clearly $E_1 \sqsubseteq E\lceil f[K]$ and $\sqsubseteq$ is transitive, hence $E_0 \sqsubseteq E\lceil f[K]$, i.e., $f[K] \notin I(E)$.

Conversely, suppose $K \in I(E_0)$ and let $\mathcal{A} = (A_n)$ be a separating family of Σ^1_1 sets for $E_0\lceil K$. Put $B_n = f[A_n]$ and $\mathcal{B} = (B_n)$. We claim that $\mathcal{B}$ is a separating family for $E\lceil f[K]$. In fact: as f is 1-1 we have that $(\forall x, y)(f(x)\, E_{\mathcal{B}}\, f(y) \leftrightarrow x\, E_{\mathcal{A}}\, y)$. Hence $(\forall z, w \in f[K])(z\, E_{\mathcal{B}}\, w \leftrightarrow zEw)$. Therefore from 2.4 we get that $f[K]$ is E-smooth.

□

Lemma 4.3 *$I(E_0)$ is not Borel. In fact we have that for every $x \in 2^\omega$ there is a continuous map $f : 2^\omega \to \mathcal{K}(2^\omega)$ such that*

(i) if γ is eventually zero, then $f(\gamma)$ is a finite subset of $[x]_{E_0}$.

(ii) if γ is not eventually zero, then $f(\gamma)$ is a non-smooth closed set (with respect to E_0).

In other words, there is a continuous reduction of $\{\alpha \in 2^\omega : \alpha$ is eventually zero $\}$ into the collection of finite subsets of $[x]_{E_0}$ and $\sim I(E_0)$. In particular $I(E_0)$ is not G_δ.

Proof: Consider the following function

$$f(\gamma) = \{\alpha \in 2^\omega : (\forall n)(\gamma(n) = 0 \to \alpha(n) = x(n))\}.$$

Clearly if γ is eventually zero, then (i) holds. On the other hand if γ has infinite many 1's, then $f(\alpha)$ is a perfect set. Let $g : 2^\omega \to 2^\omega$ be the canonical bijection of 2^ω onto $f(\gamma)$. It is not difficult to see that g is actually an embedding from E_0 into $E_0\lceil f(\gamma)$, i.e., for all $\alpha, \beta \in 2^\omega$, $\alpha E_0 \beta$ if and only if $g(\alpha)E_0 g(\beta)$. In fact: Just observe that if T is the tree of $f(\gamma)$ and some sequence in T of length n splits, then every sequence in T of length n splits.

Finally, to see that f is continuous, let for each $s \in 2^{<\omega}$

$$A_s = \{\alpha \in 2^\omega : (\forall n < lh(s))(s(n) = 0 \Rightarrow \alpha(n) = x(n))\},$$

each A_s is closed and if $t \prec s$, then $A_s \subseteq A_t$. We have that $f(\gamma) = \bigcap_n A_{\gamma\lceil n}$ and also that for every $s \in 2^{<\omega}$

$$f(\gamma) \bigcap N_s \neq \emptyset \quad \text{if and only if} \quad \forall n < lh(s)(s(n) = 0 \Rightarrow \gamma(n) = x(n))$$

which easily implies that f is continuous. By the Baire category theorem $I(E_0)$ is not G_δ and by the dichotomy theorem for σ-ideals (see [10]) $I(E_0)$ is not Borel.

□

Proof of theorem 4.1: It is clear that $I(E)$ is a σ-ideal and since the smooth sets are the common null sets of all E-ergodic, non atomic measures on X, by a standard capacitability argument we get that $I(E)$ is strongly calibrated. A similar argument as in the proof of 2.6(ii) shows that $I(E)$ is Π^1_1.

To see that $I(E)$ is locally not Borel let $K \in \mathcal{K}(X)$, we then have that

$$I(E) \cap \mathcal{K}(K) = \{F \in \mathcal{K}(K) : F \text{ is } E\text{-smooth }\} = I(E \lceil K).$$

From 4.2 we get that $I(E_0)$ is not Borel if and only if $I(E\lceil K)$ is not Borel. Now the conclusion follows from 4.3.

□

As a corollary of lemma 4.3 we get the following

Corollary 4.4 *Let E be a non smooth Borel equivalence relation on X, then*
(i) If $J \subseteq I(E_0)$ is a dense σ-ideal, then J is not $\mathbf{\Sigma^1_1}$.
(ii) If $J \subseteq I(E)$ is a σ-ideal such that for every $x \in X$ $\{x\} \in J$, then J is not $\mathbf{\Sigma^1_1}$.

Proof: (ii) follows from (i), because if $f : 2^\omega \to X$ is an embedding witnessing that E is not smooth and $J \subseteq I(E)$ is a σ-ideal containing all singletons, then $J^* = f^{-1}[J]$ is a dense σ-ideal and it is contained in $I(E_0)$.

(i) Let J be as in the hypothesis of (i). It suffices to show that J is not G_δ. Suppose toward a contradiction that $J \subseteq I(E_0)$ is a G_δ dense σ-ideal. Let $H = \{x \in 2^\omega : \{x\} \in J\}$, H is a G_δ dense set. Let G be a countable collection of homeomorphisms of 2^ω generating E_0. Put $H^* = \bigcap_{g \in G} g[H]$, H^* is an invariant dense G_δ subset of H. Let $x \in H^*$. For every y such that yE_0x, we have $\{y\} \in J$. From lemma 4.3 we get that J is not a G_δ set, a contradiction.

□

Remarks: (1) (i) above implies that there are no dense G_δ E_0-smooth sets , because if H is such a set then $\mathcal{K}(H)$ would be a dense G_δ subideal of $I(E_0)$. Actually every Baire measurable E_0-smooth set is of the first category (see [13]).

(2) (ii) above is best possible in the sense that there is a non smooth Borel equivalence relation E and a dense G_δ set H which is smooth with respect to E, hence as before we get $\mathcal{K}(H)$ is a dense Borel subideal of $I(E)$ (see [13]).

(3) Kechris (see [8]) has proved that the σ-ideal of closed sets of extended uniqueness also satisfies this hereditary property but even in a stronger form, i.e., for every perfect set M of restricted multiplicity the σ-ideal $U_0 \bigcap \mathcal{K}(M)$ has no dense $\mathbf{\Sigma^1_1}$ subideals. We do not know if this holds for $I(E_0)$.

Since for E smooth $I(E)$ is trivial, we get the following characterization of a smooth Borel equivalence relation.

Corollary 4.5 *Let E be a Borel equivalence relation on X. Then E is smooth if and only if $I(E)$ is Borel.*

□

Other structural property that has been studied in the context of σ-ideals of compact sets is the so called covering property (see [14]). This is a quite strong property and there are few known σ-ideals that have it. If for a Borel equivalence relation E it happens that $I(E)$ has the covering property then every $\mathbf{\Sigma^1_1}$ smooth sets is meager, which is not true in general. Also, from large cardinals hypothesis it can be proved that the covering property for $I(E)$ implies that every Π^1_1 E-sparse set is E-smooth (see [14]), however Becker (see [1]) has shown that in general this is not the case. It is easy to check that if $I(E_0)$ does not have the covering property then no $I(E)$ have the covering property. However, it is not known if $I(E_0)$ has this property, it will suffices to show that it has a Borel basis (see [14]). By a general fact it can be proved that $I(E_0)$does no have a $\mathbf{\Sigma}^0_2$ basis. A natural candidate for $I(E_0)$ is the collection of closed E_0-transversals (i.e. a set that meets every equivalence class in at most one point), which form a G_δ set, however it can be proved it is not a Borel basis for $I(E_0)$ (in fact it generates a not calibrated σ-ideal, see [14]).

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