NOTAS DE MATEMATICA
Nº 132

SOME RESULTS ON THE DESCRIPTIVE SET THEORY OF σ-IDEALS OF COMPACT SETS

BY

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PREPRINT

UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMATICA
MERIDA-VENEZUELA
1993

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Abstract

We present some results on the structure of σ-ideals of compact sets from a descriptive set theoretic point of view. We extend to κ-Suslin sets a theorem of Debs-Saint Raymond that gives sufficient conditions for a σ-ideal to have the covering property for analytic sets. Some results concerning the covering property and product of σ-ideals are presented.

1 Introduction.

The study of σ−ideals of compact sets has been motivated by problems in analysis and recently it has received a lot of attention because its connections with harmonic analysis (see [5]). The descriptive set theoretic approach was initiated by Kechris, Louveau and Woodin in [6] (see also [4] and [10]) where it was shown that many structural properties of σ-ideals of compact sets follow from descriptive set theoretical hypothesis. This paper continuous this line of research and it deals with the so called covering property and product of σ-ideals.

Let us fix the notation and basic notions. Throughout this article X will be a compact metric space. We will use standard notions of descriptive set theory as in Moschovakis' book [8] and we follow the notations from [6]. For instance, $\mathbf{\Sigma}^1_1$ denotes the analytic sets, i.e. the continuous images of Borel sets, and $\mathbf{\Pi}^1_1$ denotes the co-analytic sets, i.e. sets whose complements are analytic. The corresponding effective pointclasses are denoted respectively by Σ^1_1 and Π^1_1. When we work with the effective methods of descriptive set theory we assume that X is recursively presented compact metric space (see [8]).

By $\mathcal{K}(X)$ we denote the collection of closed subsets of X which becomes itself a compact, metric space under the usual Hausdorff metric:

$$\rho(K,L) = \begin{cases} \mathrm{Sup}\{\max\{d(x,K), d(y,L)\} : \ x \in K, \ y \in L\} & \text{, if } K, L \neq \emptyset \\ \mathrm{diam}(X) & \text{, if } K \text{ or } L = \emptyset \\ 0 & \text{, if } K = L = \emptyset. \end{cases}$$

All topological and descriptive set theoretic notions concerning $\mathcal{K}(X)$ refer to this space (for more details about the topology over $\mathcal{K}(X)$ see [6] and the references given there).

A subset $I \subseteq \mathcal{K}(X)$ is called **hereditary** if for all K,$L \in \mathcal{K}(X)$ with $K \in I$ and $L \subseteq K$ we have that $L \in I$. I is called an **ideal**, if moreover for all K,$L \in I$ we have taht $K \cup L \in I$. and I is called a $\sigma-$ideal, if in addition we have that if $K, K_1, K_2, \ldots \in \mathcal{K}(X)$ are such that for all i $K_i \in I$ and $K = \bigcup K_i$, then $K \in I$.

Let us give some examples:

(1) For each $A \subseteq X$, let $\mathcal{K}(A) = \{K \in \mathcal{K}(X) : \ K \subseteq A\}$.

(2) $K_\omega(X) = \{K \in \mathcal{K}(X) : \ K$ is countable $\}$.

(3) $I_{meager} = \{K \in \mathcal{K}(X) : \ K$ is meager $\}$.

(4) Given a Borel measure μ over X, let

$$I_\mu = \{K \in \mathcal{K}(X) : \ \mu(K) = 0\}$$

(5) Let R =Rajchman probability measures on the unit circle, i.e. those measures for which $\hat{\mu}(n) \to 0$, as $\mid n \mid \to \infty$. Let

$$U_0 = \{K \in \mathcal{K}(X) : \ \mu(K) = 0 \text{ for all } \mu \in R\}.$$

U_0 are the closed sets of extended uniqueness (see [5]).

(6) Let $X = 2^\omega$, and I_c= The $\sigma-$ideal of closed subsets of 2^ω that avoid a cone of Turing degrees.

Given a $\sigma-$ideal I of closed subsets of X, the most natural way to extend I to a $\sigma-$ideal of arbitrary subsets of X is as follows: Let

$$I^{ext} = \{A \subseteq X : \exists (K_n)_{n\in\omega} \text{ in } I,\ A \subseteq \bigcup_n K_n\}$$

I^{ext} is the smallest $\sigma-$ideal of subsets of X extending I. A typical example is when $I = I_{meager}$; the exterior extension of I is the $\sigma-$ideal of meager sets. Analogously the exterior extension of $K_\omega(X)$ is the $\sigma-$ideal of countable sets.

In some cases, however, the exterior extension is not the natural one. For example: if λ is the product measure on 2^ω and $I = I_\lambda$ then I^{ext} is not the $\sigma-$ideal of λ-measure zero sets. But this example suggests other way of extending I: Let

$$I^{int} = \{A \subseteq X : \mathcal{K}(A) \subseteq I\}$$

Clearly I^{int} is hereditary, $I^{ext} \subseteq I^{int}$ and $I^{int} \cap \mathcal{K}(X) = I$. But in general I^{int} is not even an ideal.

We say that a $\sigma-$ideal I on X has the **covering property** if $I^{ext} = I^{int}$ for $\mathbf{\Sigma}^1_1$ sets, i.e. a $\mathbf{\Sigma}^1_1$ set A is in I^{int} iff A is in I^{ext} (see [3], [4] and also the notion of I-regularity of [7]). This is a quite strong property, in fact the only known σ-ideals of compact sets that have the covering property are $K_\omega(X)$ and U_0. For $K_\omega(X)$, the classical perfect set theorem for $\mathbf{\Sigma}^1_1$ sets is the assertion that $K_\omega(X)$ has the covering property. And for U_0 is a theorem of Debs and Saint Raymond (see [1]). In [10] some analogies between the covering property and the perfect set theorem has been studied.

A σ-ideal I is **calibrated** if for every closed set F the following holds: If for some collection $\{F_n\}$ of closed sets in I, $F - \bigcup_n F_n \in I^{int}$, then $F \in I$. A typical calibrated σ-ideal is the collection of closed null sets with respect to some Borel measure. On the other hand, the σ-ideal of closed meager sets is not calibrated. Notice also that the covering property clearly implies calibration.

Most of the structural theory of σ-ideals has been developed imposing definibility conditions over I, for example it has to be a $\mathbf{\Pi}^1_1$ subset of $\mathcal{K}(X)$. A very important structural property of $\sigma-$ideals is the **Dichotomy theorem**: Let I be a Π^1_1 $\sigma-$ideal of compact sets, then either I is G_δ or Π^1_1-complete ([6]). Another definibility condition is the following: Let B be a hereditary subset of $\mathcal{K}(X)$, B_σ denotes the smallest $\sigma-$ideal (of closed sets) containing

B, i.e., $K \in B_\sigma$ if there is a sequence $\{K_n\}$ of elements of B such that $K = \bigcup_n K_n$. We say that I has a **Borel basis** if there is a Borel hereditary set $B \subseteq I$ such that $I = B_\sigma$. I is called **locally non-Borel** if for every closed set $F \notin I$, $I \cap \mathcal{K}(F)$ is not Borel.

The only criterion known to show that an σ-ideal has the covering property is the following theorem dued to Debs and Saint Raymond, which was originally used to show that the $\sigma-$ideal of closed set of uniqueness does not have a Borel basis (see [5] for a proof of both results).

Theorem 1.1 *(Debs-Saint Raymond [1]). Let I be a calibrated, locally non-Borel, $\mathbf{\Pi}^1_1$ σ-ideal. If I has a Borel basis, then I has the covering property.*

□

In §1 we show that the proof of 1.1 given in [5] can be extended to κ-Suslin sets. It is an open question whether or not a G_δ σ-ideal can have the covering property, in §2 we present some results concernig this question and in §3 we present some results about product of σ-ideals showing a relation between the covering property and what we call the Fubini property. The results on this article are part of my Caltech's Ph.D thesis under the supervision of Dr. Alexander Kechris to whom I am very grateful for his guidence and patience.

2 The covering property for κ-Suslin sets

Let κ be an infinite cardinal, κ^+ denote the smallest cardinal larger than κ, ω denotes the first infinite ordinal, i.e., the natural numbers. Put in κ^ω the product topology. The usual basis for this topology is given by the collection of sets $N_t = \{f \in K^\omega : f \text{ extends } t\}$, where t is a finite sequence on κ, i.e. $t \in \kappa^{<\omega}$ (see [8]). As usual for every $f \in \kappa^\omega$ and $n \in \omega$ $f\lceil n$ denotes the restriction of f to the first n values, so $f\lceil n \in \kappa^{<\omega}$.

A subset $A \subseteq X$ is called **κ-Suslin** if there is a closed $F \subseteq X \times \kappa^\omega$ such that $A = proj(F)$, i.e.,

$$x \in A \text{ iff } \exists f \in \kappa^\omega[(x, f) \in F].$$

We will write in this case $A = p[F]$. It is a classic result that analytic sets are exactly the $\aleph_0$-Suslin sets or simply Suslin sets. Most of the properties of analytic sets can be proved for κ-Suslin sets (see [8]). For instance, the perfect set theorem can be extended to κ-Suslin sets: Let A be a κ-Suslin set with more than κ elements, then A contains a non-empty perfect set. This result can be rephrased as follows: Let I be the σ-ideal of countable closed subsets of 2^ω and $A \subseteq 2^\omega$ be a κ-Suslin set in I^{int}, then $|A| \leq \kappa$, i.e., A can be covered by less than κ^+ sets in I (see [8] Thm 2C.2). In this section we will show that similarly Debs-Saint Raymond theorem 1.1 can be extended to κ-Suslin sets.

Theorem 2.1 *Let I be a $\mathbf{\Pi}^1_1$, locally non Borel, calibrated σ-ideal of closed meager subsets of X with a Borel basis. If A is a κ-Suslin set in I^{int}, then A can be covered by less that κ^+ many closed sets in I.*

The proof is based on the ideas of the proof of 1.1 given in [5]. We fix a σ-ideal I as in the hypothesis of 2.1. First we define a derivative on closed subsets of $X \times \kappa^\omega$ as follows:

Definition 2.2 *Let $F \subseteq X \times \kappa^\omega$ be a closed set and let V_s be an enumeration of an open basis for X*

$$(x, f) \in F^{(1)} \text{ iff } \forall s \forall n (x \in V_s \Rightarrow \overline{p[(V_s \times N_{f\lceil n}) \cap F]} \notin I).$$

By transfinite recursion we define $F^{(\alpha)}$ for all ordinals α.

Observe that $F^{(1)}$ is also a closed set. Hence there is an ordinal $\theta < \kappa^+$ such that $F^{(\theta)} = F^{(\theta+1)}$. We denote by $F^{(\infty)}$ this fixed point.

Lemma 2.3 *Let A be a κ-Suslin set and let $F \subseteq X \times \kappa^\omega$ be a closed set such that $A = p[F]$. If $F^\infty = \emptyset$, then $p[F]$ can be covered by less than κ^+ many closed sets in I.*

Proof: Let $\theta < \kappa^+$ be such that $F^{(\theta)} = \emptyset$. For each $(x, f) \in F$ there is $\alpha < \theta$ such that $(x, f) \in F^{(\alpha)} - F^{(\alpha+1)}$, thus there is n and s such that $x \in V_s$ and $\overline{p[(V_s \times N_{f\lceil n}) \cap F^{(\alpha)}]} \in I$. Then we have

$$p[F] \subseteq \bigcup \{\overline{p[V_s \times N_u \cap F^{(\alpha)}]} : s \in \omega \,\&\, u \in \kappa^{<\omega} \& \alpha < \theta$$

$$\& \, \overline{p[(V_s \times N_u) \cap F^{(\alpha)}]} \in I\}.$$

This proves the lemma.

□

To finish the prove of the theorem will suffice to prove the following:

Lemma 2.4 *If $F^\infty \neq \emptyset$, then $p[F] \notin I^{int}$.*

Proof: We will show that if $F \subseteq X \times \kappa^\omega \neq \emptyset$ is closed and $F^{(1)} = F$, then $p[F] \notin I^{int}$.

Let $B \subseteq I$ be a Borel basis for I. We will construct for each $t \in \omega^{<\omega}$ an element $u_t \in \kappa^{<\omega}$, an open set V_t and $K_t \in \mathcal{K}(X)$ such that

(i) $K_t \subseteq L_t = \overline{p[(V_t \times N_{u_t}) \cap F]}$ and $K_t \in I - B$.

(ii) $diam(V_{t^\frown(n)}) \leq 2^{-lh(t)}$, for all $n \in \omega$.

(iii) $V_{t^\frown(m)} \cap K_t = \emptyset$ for all $m \in \omega$.

(iv) $\overline{V_{t^\frown(n)}} \cap \overline{V_{t^\frown(m)}} = \emptyset$, for all $n \neq m$.

(v) $K_t \cup \bigcup_n L_{t^\frown(n)} = \overline{\bigcup_n L_{t^\frown(n)}}$.

(vi) $\overline{V_{t^\frown(n)}} \subseteq V_t$, $u_{t^\frown(n)}$ strictly extends u_t and $Lim_n diam(V_{t^\frown(n)}) = 0$.

For $t = \emptyset$, put $s_t = u_t = \emptyset$ and $V_\emptyset = X$, thus $L_\emptyset \notin I$. Since I is locally non Borel, there is $K_\emptyset \subseteq L_\emptyset$ such that $K_\emptyset \in I - B$.

Assume we have defined K_t, V_t and u_t for all $t \in \omega^{<\omega}$ with $lh(t) = k$. Notice that L_t is locally not in I, hence K_t is nowhere dense in L_t. It is not difficult to find (see [5] page 202, lemma 5) a countable discrete set $D_t \subseteq p[(V_t \times N_{u_t}) \cap F]$ such that

$$D_t \cap K_t = \emptyset \text{ and } K_t \cup D_t = \overline{D_t}.$$

Let $\{x_n\}$ be an enumeration of D_t. For each n find an open set $V_{t^\frown(n)}$, $u_{t^\frown(n)} \in \kappa^{<\omega}$ properly extending u_t so that

$$x_n \in p[(V_{t^\frown(n)} \times N_{u_{t^\frown(n)}}) \cap F]$$

and also

$$L_{t^\frown(n)} = \overline{p[(V_{t^\frown(n)} \times N_{u_{t^\frown(n)}}) \cap F]}$$

satisfies (ii), (iii), (iv), (v) and (vi) (for (v) observe that $diam(L_{t^\frown(n)}) \to 0$, when $n \to \infty$).

Now we want to define $K_{t^\frown(n)}$ for each n. Since $L_t \notin I$, as before we can find $K_{t^\frown(n)} \subseteq L_{t^\frown(n)} \in I - B$. Clearly all conditions (i)-(vi) are satisfied.

Claim: *Let $K = \overline{\bigcup_t K_t}$. Then $K \notin I$.*

Proof: We will show that if V is an open set and $V \cap K \neq \emptyset$ then $\overline{V \cap K} \notin B$, which says that K is locally not in I. Let V be an open set such that $V \cap K \neq \emptyset$. For some $t \in \omega^{<\omega}$, $V \cap K_t \neq \emptyset$. Since $diam(L_{t^\frown(n)}) \to 0$, when $n \to \infty$, then from (v) we get that for some n, $L_{t^\frown(n)} \subseteq V$. Thus $K_{t^\frown(n)} \subseteq V$ and in consequence $K_{t^\frown(n)} \subseteq \overline{V \cap K}$. Therefore from (i) we get that $K_{t^\frown(n)} \notin B$.

($\square$ Claim.)

As I is calibrated there is a closed set $M \subseteq K - \bigcup_t K_t$ with $M \notin I$. We will show that $M \subseteq p[F]$ and we will be done.

Put

$$F_n = \bigcup\{K_t : lh(t) < n\} \cup \bigcup\{L_t : lh(t) = n\}.$$

We claim that each F_n is closed: we show it for $n = 2$, the other cases are similar. Let $\{y_i\}$ be a sequence in F_2 and suppose that $y_i \to y$. Assume $y \notin \bigcup\{K_t : lh(t) < 2\}$, we will show that $y \in L_t$ for some t with $lh(t) = 2$. By (v) we can assume that $y_i \in L_{t_i}$ with $lh(t_i) = 2$ (or replace $\{y_i\}$ by other sequence satisfying this condition and with the same limit). From (ii) and since every D_t is a discrete set, it is easy to show that there is n such that $y_i \in L_{<n,m_i>}$ for infinite many i's. From (v) and since $y \notin K_{<n>}$, we get that $y \in L_{<n,m>}$ for some m.

From (v) we get that $K \subseteq F_n$ for every n. Therefore $M \subseteq F_n$ for every n and thus $M \subseteq \bigcap_n F_n$. Hence

$$M \subseteq \bigcap_n \bigcup_{lh(t)=n} L_t.$$

From this and (vi) it is easy to see that $M \subseteq p[F]$.

$\square$

3 On the covering property for σ-ideals of compact sets

The covering property is an abstract version of the classical perfect set theorem (see [10]). In this section we will present some results concerning this property. As we said it is not known whether or not a Borel (i.e, G_δ) σ-ideal can have the covering property. We will give a partial answer to that question.

Lemma 3.1 *Let $D \subseteq \mathcal{K}(X)$ be an open hereditay set such that if $F \in D$ and $x \in X$, then $F \cup \{x\} \in D$. Then there is an open dense set U such that $\mathcal{K}(U) \subseteq D$.*

Proof: Let $\{x_n\}$ be a countable dense subset of X. We will define a sequence $\{O_n\}$ of open sets such that $x_n \in O_n$ and $\bigcup_{j=1}^{N} \overline{O_n} \in D$, for each N.

First, observe that if $F \in D$, then there is an open set O such that $F \subseteq O$ and $\mathcal{K}(O) \subseteq D$. To see this, note that since D is open, there is an open nghd W in $\mathcal{K}(X)$, such that $F \in W$ and $W \subseteq D$. Say $W = \{K \in \mathcal{K}(X) : K \subseteq V_0 \& K \cap V_i \neq \emptyset, 1 \leq i \leq n\}$, where each V_i is an open subset of X. We claim that $\mathcal{K}(V_0) \subseteq D$: if $K \subseteq V_0$, let $y_i \in V_i$ for $1 \leq i \leq n$; then $K \cup \{y_i : 1 \leq i \leq n\} \in W$, hence $K \cup \{y_i : 1 \leq i \leq n\} \in D$. But as D is hereditary, then $K \in D$.

We define $\{O_n\}$ by induction on n. For $n = 0$: as $\{x_0\} \in D$, there is an open set O such that $x_0 \in O$ and $\mathcal{K}(O) \subseteq D$. Let O_0 be an open set such that $x_0 \in O_0$ and $\overline{O_0} \subseteq O$.

Suppose we have defined O_n for $0 \leq n \leq N$ such that $x_n \in O_n$ and $\bigcup_{j=0}^{N} \overline{O_j} \in D$. Then by hypothesis $\bigcup_{j=0}^{N} \overline{O_j} \cup \{x_{N+1}\} \in D$. By the observation above, there is an open set V such that $\bigcup_{j=0}^{N} \overline{O_j} \cup \{x_{N+1}\} \subseteq V$ and $\mathcal{K}(V) \subseteq D$. Let O_{N+1} be an open set such that $x_{N+1} \in O_{N+1}$ and $\overline{O_{N+1}} \subseteq V$. Clearly $\bigcup_{j=0}^{N+1} \overline{O_j} \in D$.

Finally, put $U = \bigcup_{j=0}^{\infty} O_j$. U is clearly an open dense set. Now, if $F \subseteq U$, by compactness, there is N such that $F \subseteq \bigcup_{j=0}^{N} O_j \subseteq \bigcup_{j=0}^{N} \overline{O_j}$. Since D is hereditary $F \in D$, i.e., $\mathcal{K}(U) \subseteq D$.

□

Theorem 3.2 *Let I be a $\mathbf{\Pi}_2^0$ hereditary collection of compact sets. Assume there are open sets D_n in $\mathcal{K}(X)$ such that $I = \bigcap_n D_n$ and for all $F \in D$ and all $x \in X$ we have $F \cup \{x\} \in D_n$. Then there is a dense G_δ set G such that $\mathcal{K}(G) \subseteq D$, i.e., $G \in I^{int}$. In particular, if I is a $\mathbf{\Pi}_2^0$ ideal of closed meager sets as above, then I does not have the covering property for $\mathbf{\Pi}_2^0$ sets.*

Proof: First, we can assume that each D_n is hereditary. In fact, consider *the following sets:*

$$J_n = \{K \in \mathcal{K}(X) : \forall F(F \subseteq K \rightarrow F \in D_n)\}.$$

Recall that the relation $R(F, K)$ iff $F \subseteq K$ is closed in $\mathcal{K}(X) \times \mathcal{K}(X)$. Thus J_n is open (recall that the proyection of an F_σ set in a compact space is F_σ) and it is clearly a hereditary subset of D_n. Notice also that if $F \in J_n$ and $x \in X$, then $F \cup \{x\} \in J_n$. Now, as I is hereditary if $F \in I$, then $F \in J_n$ for all n, i.e., $I = \cap_n J_n$.

To prove the theorem, we have by the previous lemma that there are open dense sets O_n such that $\mathcal{K}(O_n) \subseteq D_n$. Put $G = \bigcap_n O_n$. G is a dense G_δ in I^{int}.

Finally, the Baire category theorem implies that such G can not be covered by countable many meager closed sets, in particular $G \notin I^{ext}$.

□

Remark: We do not know of any $\mathbf{\Pi}^0_2$ ideal which does not satisfy the hypothesis of the previous theorem, even in the following weaker form: there is a dense countable set D such that the condition about $\{x\} \cup F$ holds only for $x \in D$.

Another notion related with the covering property is thinnes. We say that I is *thin* if every collection of pairwise disjoint closed sets not in I is at most countable. The typical example of thin ideal is the collection of null closed set for some Borel measure. The following theorem relates thinnes with the covering property.

Theorem 3.3 *([10] Theorem 2.5) Let I be a σ-ideal of closed sets which satisfies one of the following non triviality conditions:*

(i) $I \neq \mathcal{K}(X)$ and for every $x \in X$, $\{x\} \in I$.

(ii) Every $K \in I$ is a meager set.

If I is thin, then I does not have the covering property for $\mathbf{\Pi}^0_2$ sets. Actually, if (ii) holds, then there is a dense G_δ set in I^{int}.

Theorem 3.3 connects the covering property with the notion of controlled ideal. Let's recall this notion. Let $G \subseteq 2^\omega \times X$ be a Π^0_2-universal set for $\mathbf{\Pi}^0_2$ subsets of X. A code for a $\mathbf{\Pi}^0_2$ set H is an $\alpha \in 2^\omega$ such that $H = G_\alpha$. A collection A of $\mathbf{\Pi}^0_2$ subsets of X is *compatible* with I if the least $\sigma-$ideal J of $\mathbf{\Pi}^0_2$ sets containing I and A extends I, i.e., it satisfies $J \cap \mathcal{K}(X) = I$. An ideal I is said to be *controlled* if there is a $A \subseteq \mathbf{\Pi}^0_2(X)$ such that $\emptyset \in A$, A is compatible with I and A is Σ^1_1 in the codes of $\mathbf{\Pi}^0_2$ sets (i.e., $\{\alpha \in 2^\omega : G_\alpha \in A\}$ is Σ^1_1). Such set A is called a *control set* for I.

Observe that for a calibrated $\sigma-$ideal I, A is compatible with I iff $A \subseteq I^{int} \cap \mathbf{\Pi}^0_2(X)$. The following theorem was proved in [6].

Theorem (Kechris, Louveau, Woodin, see [6]): *Let I be a controlled Π^1_1 σ-ideal of closed subsets of X. Then I is $\mathbf{\Pi}^0_2$ and thin.* □

From this and 3.3 we immediately get the following

Corollary 3.4 *Let I be a Π^1_1 non trivial σ-ideal in the sense of 3.3. If I has the covering property for $\mathbf{\Pi}^0_2$ sets, then I is not controlled.* $\square$

We do not know yet if there are $\mathbf{\Pi}^0_2$ $\sigma-$ideals with the covering property. However, the corollary above implies that every non trivial Π^1_1 $\sigma-$ideal of closed sets with the covering property has to be true Π^1_1 on the codes of $\mathbf{\Pi}^0_2$ sets. This will follow from the following lemma:

Lemma 3.5 *Let G be a Π^0_2 universal sets for $\mathbf{\Pi}^0_2$ subsets of X and I a Π^1_1 σ-ideal of closed subsets of X. Then*

(i) $\{\alpha \in 2^\omega : G_\alpha \in I^{int}\}$ is Π^1_1.

(ii) $\{\alpha \in 2^\omega : G_\alpha$ is closed $\}$ is Π^1_1.

Proof: (i) First, we have that

$$G_\alpha \in I^{int} \text{ iff } \forall F \in \mathcal{K}(X)(F \subseteq G_\alpha \Rightarrow F \in I)$$

Now, the relation $R(F, \alpha) \Leftrightarrow F \subseteq G_\alpha$ is Π^0_2, because

$$F \subseteq G_\alpha \text{ iff } \forall x(x \notin F \text{ or } (\alpha, x) \in G).$$

And recall that the projection of a F_σ subset of a compact space is F_σ.

(ii) Fix a countable open basis for the topology of X, say $\{V_n : n \in \mathbf{N}\}$. Then

$$G_\alpha \text{ is closed iff } (\forall x)[(\forall n)(x \in V_n \Rightarrow V_n \cap G_\alpha \neq \emptyset) \Rightarrow x \in G_\alpha]. \qquad (*)$$

Now, the following relation is clearly Σ^1_1.

$$R(n, \alpha, x) \text{ iff } (x \in V_n \Rightarrow V_n \cap G_\alpha \neq \emptyset) \text{ iff } x \notin V_n \text{ or } (\exists y)(y \in V_n \ \& \ (\alpha, y) \in G).$$

Hence (*) is Π^1_1. $\square$

Proposition 3.6 *Let I be a Π^1_1 σ-ideal of closed subsets of X, which is non trivial in the sense of 3.3. If I has the covering property, then $\{\alpha \in 2^\omega : G_\alpha$ is closed and $G_\alpha \in I\}$ is a true Π^1_1 set.*

Proof: Let $A = \{G_\alpha : G_\alpha$ is closed and $G_\alpha \in I\}$, then $\emptyset \in A$ and $A \subseteq \mathbf{\Pi}^0_2(X) \cap I^{int}$. As I is not controlled (by 3.4), then A is not Σ^1_1 on the codes of $\mathbf{\Pi}^0_2$ sets. Hence from 3.5 we get the conclusion of the proposition. $\square$

Proof: Let $B = \bigcup_m L_m$ be a basis for I, with each L_m a closed set. Since $Her(L_m) = \{K : \exists F \in L_m$ such that $K \subseteq F\}$ is also a closed subset of I, we can assume without loss of generality that each L_m is hereditary. Also assume that $L_m \subseteq L_{m+1}$.

We claim that each L_m is meager: Suppose, towards a contradiction, that $W \subseteq L_m$ is an open set. As L_m is hereditary there is an open set $V \subseteq X$ such that $\mathcal{K}(V) \subseteq L_m$, which contradicts that every set in I is meager.

Fix a dense set $D \subseteq X$. We will define a sequence F_s for $s \in 2^{<\omega}$ such that

(1) F_s is a finite subset of D.

(2) If $s \prec t$, then $F_s \subseteq F_t$ and $dist(F_s, F_t) \leq 1/2^{lh(s)}$.

(3) For all $x \in F_s$ there is $K_x^s \notin L_{lh(s)}$ such that $K_x^s \subseteq F_{s\frown(1)}$ and $diam(K_x^s) \leq 1/2^{lh(s)+2}$.

(4) $F_{s\frown(0)} = F_s$.

Assuming this sequence has been defined we finish the proof. Put

$$f(\alpha) = \overline{\bigcup_n F_{\alpha\lceil n}}.$$

By the previous lemma we have that

$$f(\alpha) = lim_n F_{\alpha\lceil n}.$$

This clearly implies that f is continuous: In fact, we easily get that if $\alpha\lceil n = \beta\lceil n$, then $dist(F_{\alpha\lceil m}, F_{\beta\lceil m}) \leq 2/2^n$ for all $m > n$.

By (4), it is clear that if α is enventually zero, then $f(\alpha)$ is a finite subset of D. Now, suppose that α has infinite many 1's. We will show that $f(\alpha)$ is locally not in I. Put $F = f(\alpha)$. Let V be an open subset of X with $F \cap V \neq \emptyset$. Then there is n such that $F_{\alpha\lceil n} \cap V \neq \emptyset$. Let $x \in F_{\alpha\lceil n} \cap V$, thus $x \in F_{\alpha\lceil m} \cap V$, for all $m \geq n$. As $diam(K_x^{\alpha\lceil m}) \to 0$, then there is N such that for all $m \geq N$,

$$K_x^{\alpha\lceil m} \subseteq V \cap F_{\alpha\lceil m} \subseteq V \cap F.$$

Therefore for all $m \geq N$ $\overline{V \cap F} \notin L_m$, which implies that $\overline{V \cap F} \notin I$.

We define the sequence F_s by induction on the length of $s \in 2^{<\omega}$. Fix $x_0 \in D$ and let $F_\emptyset = \{x_0\}$. Suppose we have defined F_s for all $s \in 2^n$ and (1)-(4) are satisfied. Put $F_{s\frown(0)} = F_s$. To define $F_{s\frown(1)}$ consider the following: For every $x \in F_s$ let V_x^s be an open ball such that $x \in V_x^s$ and $diam(V_x^s) \leq$

$1/2^{lh(s)+2}$. As $L_{lh(s)}$ is meager, then there is $T^s_x \subseteq V^s_x$ such that $T^s_x \notin L_{lh(s)}$. As D is dense there is $K^s_x \subseteq D$ finite such that $K^s_x \subseteq V^s_x$. Now, one of those K^s_x's is not in $L_{lh(s)}$: Otherwise, as $L_{lh(s)}$ is closed, then T^s_x would be in $L_{lh(s)}$.

So put

$$F_{s^\frown(1)} = F_s \cup \{K^s_x : x \in F_s\}.$$

Notice, for every $y \in F_{s^\frown(1)}$ there is $x \in F_s$ such that $y \in K^s_x \cup F_s$ and $d(x,y) \leq 1/2^{lh(s)+1}$. Hence $dist(y, F_s) \leq 1/2^{lh(s)+1}$.

Thus $F_{s^\frown(1)}$ satisfies (1)-(4). This finishes the construction of f.

To finish the proof of the theorem, let $J \subseteq I$ be a dense $\sigma-$ideal. We will show that J is not Borel. By the dichotomy theorem it suffices to show that J is not $\mathbf{\Pi}^0_2$. Let $D = \{x \in X : \{x\} \in J\}$. As J is dense, so is D. We just have proved that there is a continuous reduction of the eventually zero sequences into the collection of finite subsets of D and the complement of I. In particular it says that we cannot separate with a G_δ set the collection of finite subsets of D from the complement of J. Hence J is not $\mathbf{\Pi}^0_2$.

Finally, let F be a closed set locally not in I and I' be the restriction of I to $\mathcal{K}(F)$. I' clearly has a $\mathbf{\Sigma}^0_2$ basis and since F is locally not in I, then every set in I' is meager in F. Hence the same argument applies.

□

As we have said before A. Louveau has given a more general argument: Let I be a $\mathbf{\Pi}^1_1$ dense $\sigma-$ideal of closed meager sets which is meager (as a subset of $\mathcal{K}(X)$). For every dense set $D \subseteq X$ there is a continuous function $f : 2^\omega \to \mathcal{K}(X)$ as in the statement of the previous theorem and such if α is eventually zero, then $f(\alpha)$ is a finite subset of D. In particular, if $J \subseteq I$ is a dense $\sigma-$ideal then J is not Borel.

Let D be a countable dense subset of X such that for all $x \in D$ $\{x\} \in I$. Let $G \subseteq \mathcal{K}(X)$ be a G_δ dense set such that $I \cap G = \emptyset$. Put $A = \{F \in \mathcal{K}(X) : F$ is a finite subset of $D\}$. A is a dense F_σ set. By the Baire category theorem no F_σ set L separates G from A (i.e., $G \subseteq L$ and $L \cap A = \emptyset$). Hence by the Hurewicz-type theorem (see [6] §1 theorem 4) there is a continuous function $f : 2^\omega \to \mathcal{K}(X)$ such that

(i) If α is eventually zero, then $f(\alpha) \in A$.

(ii) If α is not eventually zero, then $f(\alpha) \in G$.

This function clearly works. Let us observe that if I has a $\mathbf{\Sigma}^0_2$ basis, then the collection of I-perfect sets is a $\mathbf{\Pi}^0_2$ dense set. Hence I is meager.

To see how the covering property follows from a definability condition suppose I is a $\sigma-$ideal which does not have non-trivial dense Borel subideal and suppose also that this holds locally i.e., if M is locally not in I, then $I \cap \mathcal{K}(M)$ does not have non-trivial dense (in $\mathcal{K}(M)$) Borel subideal. In particular, if $G \subseteq X$ is G_δ dense set, then $\mathcal{K}(G) \not\subseteq I$ i.e., $G \notin I^{int}$ and the same happens locally. That is to say, I has the covering property for $\mathbf{\Pi}^0_2$ sets. By the theorem 3.8 this is the case of a $\sigma-$ideal I with a $\mathbf{\Sigma}^0_2$ basis, in fact in [4] it was shown that such I has the covering property.

4 Products of σ-ideals

Given two $\sigma-$ideals I and J over X and Y respectively there is a natural way to define an ideal $I \times J$ over $X \times Y$ (see the definition below). In this section we will look at $I \times J$ from a descriptive set theoretic point of view. We will prove that if I and J are Π^1_1 $\sigma-$ideals then so is $I \times J$ (assuming some extra hypothesis on I). In fact, if I and J are G_δ $\sigma-$ideals (recall the Dichotomy theorem) then $I \times J$ is also G_δ. We will also see some results on the covering property and about the Fubini theorem in this abstract setting of $\sigma-$ideals of compact sets.

Definition 4.1 *Let X and Y be compact Polish spaces. Let I and J be σ-ideals on X and Y respectively. Define the product of I and J as follows: Let $K \subseteq X \times Y$ be a closed set, denote by K_x the x-section of K, i.e., $K_x = \{y \in Y : (x,y) \in K\}$*

$$K \in I \times J \ \textit{ iff } \ \{x \in X : K_x \notin J\} \in I^{int}.$$

If J is $\mathbf{\Pi}^0_2$, then for every closed subset K of $X \times Y$ $\{x : K_x \notin J\}$ is $\mathbf{\Sigma}^0_2$. So $\{x : K_x \notin J\} = \bigcup_n F_n$ for some closed sets F_n. Then $K \in I \times J$ iff for all n, $F_n \in I$. We will see below that if I is also $\mathbf{\Pi}^0_2$, then $I \times J$ is a $\mathbf{\Pi}^0_2$ $\sigma-$ideal.

On the other hand if J is $\mathbf{\Pi}^1_1$, then $\{x : K_x \notin J\}$ is $\mathbf{\Sigma}^1_1$. So, in order to get that $I \times J$ is a $\sigma-$ideal we need that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} forms a $\sigma-$ideal. This happens, for instance, when I is strongly calibrated (see lemma 3.6 of [10]). We will show that under this hypothesis we also get that $I \times J$ is a $\mathbf{\Pi}^1_1$ calibrated $\sigma-$ideal.

Proposition 4.2 *Let I and J be $\mathbf{\Pi}^0_2$ σ-ideals of closed subsets of X and Y respectively. Then $I \times J$ is a $\mathbf{\Pi}^0_2$ σ-ideal of closed subset of $\mathcal{K}(X) \times \mathcal{K}(Y)$.*

Proof: Consider the following relation on $X \times \mathcal{K}(X \times Y)$

$$P_J(x, K) \Longleftrightarrow K_x \in J.$$

Claim: P_J is $\mathbf{\Pi}^0_2$.
Proof: We have that

$$P_J(x, K) \Longleftrightarrow \forall L \in \mathcal{K}(Y)\, [L \subseteq K_x \Rightarrow L \in J].$$

Now, consider the relation: $R(x, K, L) \Leftrightarrow L \subseteq K_x$. Then

$$R(x, K, L) \Longleftrightarrow \forall V \text{open in } Y\, [K_x \subseteq V \Rightarrow L \subseteq \overline{V}].$$

For every open set V let $R_V(L) \Leftrightarrow L \subseteq \overline{V}$ and $R'_V(x, K) \Leftrightarrow K_x \subseteq V$. Clearly R_V is closed in $\mathcal{K}(Y)$ and

$$R'_V(x, K) \Longleftrightarrow (\forall y \in Y)[(x, y) \in K \Leftrightarrow y \in V].$$

Thus the complement of R'_V is the projection of a compact set. Hence R'_V is open. Therefore R is closed and thus P_J is $\mathbf{\Pi}^0_2$. ($\Box$ Claim)

From the claim there are closed sets $F_n \subseteq X \times \mathcal{K}(X \times Y)$ such that

$$\{(x, K) : \ K_x \notin J\}\{= \bigcup_n F_n$$

For each $K \subseteq X \times \mathcal{K}(X \times Y)$ closed let $F_n(K) = \{x \in X : (x, K) \in F_n\}$. Notice that $F_n(K)$ is closed.

Then

$$\begin{aligned} K \in I \times J \quad & \text{iff} \quad \{x : K_x \notin J\} \in I^{int} \\ & \text{iff} \quad \textstyle\bigcup_n F_n(K) \in I^{int} \\ & \text{iff} \quad \forall n[F_n(K) \in I]. \end{aligned}$$

As before we have that $\{K \in \mathcal{K}(X \times Y) : F_n(K) \in I\}$ is $\mathbf{\Pi}^0_2$. Therefore $I \times J$ is $\mathbf{\Pi}^0_2$.

It is clear that $I \times J$ is hereditary. To see that $I \times J$ is a σ-ideal let $K = \bigcup K_n$ be a closed set with each $K_n \in \mathcal{K}(X \times Y)$. As before we get that

$$\{x : K_x \notin J\} = \bigcup_m \{x : (K_m)_x \notin J\} = \bigcup_{n,m} F_n(K_m).$$

Thus

$$\begin{aligned} K \in I \times J \quad & \text{iff} \quad \forall n\, \forall m\, F_n(K_m) \in I \\ & \text{iff} \quad \forall m\, K_m \in I \times J. \end{aligned}$$

$\square$

As we said before in the case that I and J are $\mathbf{\Pi}^1_1$ we need an extra hypothesis to get a similar result as in 4.2. Let recall the notion of strong calibration introduced in [6] and proved to imply calibration.

Definition 4.3 *An ideal I is* **strongly calibrated** *if for every closed set $F \subseteq X$ with $F \notin I$ and every $\mathbf{\Pi}^0_2$ set $H \subseteq X \times 2^\omega$ such that $proj(H) = F$, there is a closed set $K \subseteq H$ such that $proj(K) \notin I$.*

Proposition 4.4 *Suppose I is a strongly calibrated $\mathbf{\Pi}^1_1$ σ-ideal on X and J a $\mathbf{\Pi}^1_1$ calibrated σ-ideal on Y. Then $I \times J$ is a calibrated $\mathbf{\Pi}^1_1$ σ-ideal on $X \times Y$.*

Proof: For every $K \in \mathcal{K}(X \times Y)$ $\{x : K_x \notin J\}$ is a $\mathbf{\Sigma}^1_1$ set. By lemma 3.6 of [10] we know that the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is a $\sigma-$ideal. From this we easily get that $I \times J$ is a $\sigma-$ideal.

To show that $I \times J$ is $\mathbf{\Pi}^1_1$ consider the following relation: Let $Q \subseteq \mathcal{K}(Y) \times 2^\omega$ be a $\mathbf{\Pi}^0_2$ set such that

$$F \notin J \text{ iff } \exists \alpha\, Q(F, \alpha).$$

Then given $K \in \mathcal{K}(X \times Y)$ and $x \in X$ we have

$$K_x \notin J \;\text{ iff }\; \exists \alpha\, \exists F\, (F = K_x \;\&\; Q(F, \alpha)).$$

So consider the following relation on $X \times \mathcal{K}(Y) \times 2^\omega \times \mathcal{K}(X \times Y)$

$$R(x, F, \alpha, K) \Leftrightarrow F = K_x \;\&\; Q(F, \alpha).$$

It is easy to check that R is $\mathbf{\Pi}^0_2$. We get

$$\{x : K_x \notin J\} = \{x : \exists \alpha \exists F\, [R(x, F, \alpha, K)]\}.$$

Since I is strongly calibrated we get

$$\{x : K_x \notin J\} \notin I^{int} \text{ iff } \exists P \in \mathcal{K}(X \times \mathcal{K}(Y) \times 2^\omega)[proj(P) \notin I \;\&\; P \subseteq R_K]$$

where

$$R_K = \{(x, F, \alpha) \in X \times \mathcal{K}(Y) \times 2^\omega : R(x, F, \alpha, K)\}.$$

And we have

$$P \subseteq R_K \text{ iff } \forall x \in X \; \forall F \in \mathcal{K}(Y) \; \forall \alpha \in 2^\omega \; [(x, F, \alpha) \in P \Rightarrow R(x, F, \alpha, K)]$$

which clearly is a $\mathbf{\Pi}^0_2$ relation on P and K. Hence $\{x : K_x \notin J\} \notin I^{int}$ is a $\mathbf{\Sigma}^1_1$ relation on K, i.e., $I \times J$ is $\mathbf{\Pi}^1_1$.

It remains to show that $I \times J$ is calibrated. We will need the following

Claim: Let $G \subseteq X \times Y$ be a $\mathbf{\Pi}^0_2$ set. Then $G \in (I \times J)^{int}$ iff $\{x : G_x \notin J^{int}\} \in I^{int}$.

Proof: First suppose $\{x : G_x \notin J^{int}\} \in I^{int}$. Let $K \subseteq G$ be a closed set. Then

$$\{x : K_x \notin J\} \subseteq \{x : G_x \notin J^{int}\}$$

hence $K \in I \times J$, i.e., $G \in (I \times J)^{int}$.

Conversely, suppose $\{x : G_x \notin J^{int}\} \notin I^{int}$ and let $H \subseteq \{x : G_x \notin J^{int}\}$ with $H \notin I$. Consider the following relation on $X \times \mathcal{K}(Y)$

$$R(x, F) \Leftrightarrow F \subseteq G_x \;\&\; F \notin J \;\&\; x \in H.$$

R is $\mathbf{\Sigma}^1_1$ and $proj(R) = H$. As I is strongly calibrated there is a closed $Q \subseteq R$ such that $proj(Q) \notin I$. Define $P \subseteq X \times Y$ as follows

$$P(x, y) \Leftrightarrow \exists F \in \mathcal{K}(Y) \; (y \in F \;\&\; (x, y) \in Q)$$

As $Q \subseteq R$ then P is a (closed) subset of G and $proj(Q) = \{x : P_x \notin J\} \notin I$. Hence $P \notin I \times J$, i.e., $G \notin (I \times J)^{int}$. (Claim $\square$)

Let $K = G \cup \bigcup_n H_n$ be a closed set , where $G \in (I \times J)^{int}$ is $\mathbf{\Pi}^0_2$ and each H_n is in $I \times J$. We want to show that $K \in I \times J$. For all x we have

$$K_x = G_x \cup \bigcup_n (H_n)_x.$$

Since J is calibrated one easily gets that

$$K_x \notin J \text{ iff } G_x \notin J^{int} \text{ or } \exists n \; [(H_n)_x \notin J]$$

That is to say

$$\{x : K_x \notin J\} = \{x : G_x \notin J^{int}\} \cup \bigcup_n \{x : (H_n)_x \notin J\}.$$

By the claim $\{x : G_x \notin J^{int}\} \in I^{int}$ and since every $H_n \in I \times J$ then $\{x : (H_n)_x \notin J\} \in I^{int}$. As I is strongly calibrated, the collection of $\mathbf{\Sigma}^1_1$ sets in I^{int} is a $\sigma-$ideal. So we get $\{x : K_x \notin J\} \in I^{int}$, i.e., $K \in I \times J$.

□

In relation with the covering property we have the following

Proposition 4.5 *Let I and J be σ-ideals of meager closed sets on X and Y respectively. If $I \times J$ has the covering property for $\mathbf{\Pi}^0_2$ sets, then I and J has the covering property for $\mathbf{\Pi}^0_2$ sets.*

Proof: Suppose I does not have the covering property for $\mathbf{\Pi}^0_2$ sets. By lemma 2.4 of [10] there is a locally non in I closed set M and a $\mathbf{\Pi}^0_2$ set G with $\overline{G} = M$ and $G \in I^{int}$. Put $H = G \times Y$. Clearly H is a $\mathbf{\Pi}^0_2$ set and $H \in (I \times J)^{int}$ (if $K \subseteq H$, then $\{x : K_x \notin J\} = G$). Also $\overline{H} = M \times Y$. So, it remains to show that $\overline{H}$ is locally not in $I \times J$. Let $V \subseteq X$,$W \subseteq Y$ be open sets. Then $(V \times W) \cap H = (V \cap G) \times W$. Thus

$$\{x : [\overline{(V \times W) \cap H}]_x \notin J\} = \{x : [(\overline{V} \cap M) \times \overline{W}]_x \notin J\} = \overline{V} \cap M \notin I$$

(since for every open set W, $\overline{W} \notin J$).

Analogously, if J does not have the covering property, then a similar argument shows that $I \times J$ does not have the covering property.

□

Given two ideals I and J on X there is a natural question regarding the definition of $I \times J$: Let $K \subseteq X \times X$ be a closed set, does the following hold:

$$\{x : K_x \notin J\} \in I^{int} \text{ iff } \{y : K_y \notin I\} \in J^{int} \tag{*}$$

In other words is $I \times J = J \times I$?

In particular if $I = J$ we say that I has the **Fubini property** if (*) holds for every closed $K \subseteq X \times X$. For instance, if $I = Null(\mu)$ for a measure μ on X then Fubini theorem says that I has the Fubini property. Also, if I is the ideal of meager sets, the Kuratowski- Ulam theorem (see [9]) implies that I has the Fubini property. In relation with this property we have the following

Proposition 4.6 *Let I be a $\mathbf{\Pi}_1^1$ σ-ideal of closed subsets of 2^ω. If I is not thin, then I does not have the Fubini property. In particular, if I has the Fubini property and is non trivial in the sense of 3.3, then I does not have the covering property for $\mathbf{\Pi}_2^0$ sets.*

Proof: By theorem 2 §3 [6], as I is not thin, there is a continuous function $f : 2^\omega \to \mathcal{K}(2^\omega)$ such that

(i) For all $\alpha \in 2^\omega$ $f(\alpha) \notin I$.

(ii) For all $\alpha, \beta \in 2^\omega$, if $\alpha \neq \beta$ then $f(\alpha) \cap f(\beta) = \emptyset$.

Consider the following subset of $2^\omega \times 2^\omega$

$$K(\alpha, \beta) \text{ iff } \alpha \in f(\beta)$$

then

$$K(\alpha, \beta) \text{ iff } (\exists F)(\alpha \in F \& f(\beta) = F)$$

As f is continuous then K is closed and we have that

$$\{\beta : K^\beta \notin I\} = 2^\omega \text{ and } \{\alpha : K_\alpha \notin I\} = \emptyset.$$

Hence I does not have the Fubini property. The last part of the proposition follows directly from 3.3.

□

Remark: For an arbitrary compact Polish space X we can analogously get that there is a Borel set $B \subseteq X \times X$ such that $\{\beta : B^\beta \notin I^{int}\} = 2^\omega$ and $\{\alpha : B_\alpha \notin I\} = \emptyset$ (but actually every section B^β and B_α is closed). The reason is that in this case the thickness witness $f : X \to \mathcal{K}(X)$ is a Borel function.

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