Abductive Change Operators

Jorge Lobo * Department of EECS University of Illinois at Chicago 851 S. Morgan St, Room 1120 SEO Chicago, Illinois 60607-7053, USA jorge@eecs.uic.edu Carlos Uzcátegui[†] Departamento de Matemáticas Facultad de Ciencias Universidad de Los Andes Mérida, VENEZUELA uzca@ciens.ula.ve

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Abstract

This paper describes a change theory based on abductive reasoning. We take the AGM postulates for revisions, expansions and contractions, and Katsuno and Mendelzon postulates for updates and incorporate abduction into them. A key feature of the theory is that presents a unified view of standard change operators and abductive change operators rather than a new and independent change theory for abductive changes. Abductive operators reduce to standard change operators in the limiting cases.

1 Introduction

Many actions taken by rational agents can be explained based on a cause–effect reasoning of the agent. There are actions that are taken to produce an effect. For example, we will put money in a parking-meter to avoid paying a fine to the city. Feeding the meter causes not getting a parking ticket as an effect. There are also actions that occur after an explanation or cause is derived from an observation or effect. The process of finding explanations from observations is usually referred to as abduction. Consider, for example, the following knowledge base

$$\Phi = \begin{cases} r \to w \\ p \to w \\ r \to u \\ \neg p \end{cases}$$

representing a piece of the description of Alberto's world. The propositional letter r can be read as *it is raining today*, the letter p as *the sprinkler is on*, the letter w as *the clothes are wet* and the letter u as *he should bring the umbrella with him today*. Since he himself put the sprinkler away last night, he knows that the sprinkler is not on.¹ This condition is expressed in the last sentence

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¹Actually, he assumes that the sprinkler is away, and although exceptions to this assumption can arise the problem of dealing with exceptions is outside the scope of this paper.

in Φ . Suppose his girlfriend has arrived this morning to give him a ride to work and when she came in the house he notices that her coat was wet. With this information Alberto revises his knowledge base and asserts that it must be raining and he should bring an umbrella. His reasoning was somehow abductive. From the first two formulas he could have concluded that either it was raining this morning or the sprinkler was on. But from the last sentence in Φ he was able to disregard the sprinkler and assert that it must be raining (i.e. that r is true), and hence, he will bring the umbrella.

Since the rationality postulates for belief revision proposed in [1] by Alchourrón, Gärdenfors and Makinson appeared much research has been done on how these postulates relate to different revision algorithms for knowledge bases. The AGM postulates, as they are referred to in the literature, are important for two reasons. First, the postulates are written independently of the concrete representation of the knowledge. They only assume that the knowledge base (KB) represents a closed set of sentences.² Secondly, and more important, they provide a minimal set of rational criteria to evaluate and compare change operators. However, the AGM postulates cannot be used to explain Alberto's reasoning in the example since any revision algorithm that follows the AGM postulates will simply add w to Φ and Alberto will not realize that he should bring the umbrella.

Katsuno and Mendelzon [14, 15] studied the revision theory of KBs represented by sets of propositional sentences. In their study they found that the AGM postulates did not apply to update operators as usually defined in KBs and proposed an alternative set of postulates that differentiates between updates and revision. If we apply updates to Alberto's problem we will not obtain the correct result either since according to Katsuno and Mendelzon's postulates for updates, whenever a knowledge base is updated with a formula that is consistent with the current knowledge base the resulting knowledge base must be a theory that is implied by the formula plus the original theory. Thus, under these restrictions neither r nor u will be part of the final theory.

Assume now that we attempt to model Alberto's reasoning using abduction.³ In this situation we need to select a subset of Φ as his domain theory. This is the theory that he extends with explanations to justify his observations. Assume his domain theory comprises the first three formulas in Φ . Let us call this set Σ . We also need to decide which formulas in his language can be used as explanations. Natural choices are formulas formed with the letters r and p.

The problem with this setting is that abduction will ignore the fact that $\neg p$ was true in Φ . The cautious explanation (i.e., the disjunction of all abductive explanations) for w in Σ will be $p \lor r$ and thus, the new Φ will be $\Sigma \cup \{p \lor r\}$. Any other explanation, such as r or p or $r \land \neg p$, will require assumptions that are not part of Σ . Here there is no definite answer to the question of whether to bring the umbrella to work or not.

If we assume for a moment that Σ is equal to Φ it seems that the problem can be solved since from the explanation $p \lor r$ and $\neg p$ in Σ he can deduce r and then u. However, taking the whole Φ as the domain theory will induce other problems. Take, for example, the following simpler Φ ,

$$\Phi = \begin{cases} r \to u \\ r \to u \\ \neg r \end{cases}$$

Alberto now lives in an apartment and he doesn't know about sprinklers and explanations are made with r only. He came home yesterday night and listened to the weather report that predicted a sunny day today. He wakes up thinking that it is not raining. His girlfriend arrives and her coat

 $^{^{2}}$ We will work with a propositional language, where most of the research on theory change has focused, even though the original paper of Alchourrón, Gärdenfors and Makinson was written for an unspecified language.

 $^{^{3}}$ In this paper we follow the logic-based definition of abduction as presented, for example, in [16].

is wet, but now he is not able to conclude r if $\Sigma = \Phi$ since it will make his theory inconsistent and abduction preserves consistency. On the other hand, if Alberto is able to revise the theory with the explanation r he will bring his theory up to date and avoid inconsistencies replacing $\neg r$ with r.

The major contribution of this paper is a theory of abductive changes which expands the AGM postulates for revisions, expansions and contractions, and Katsuno and Mendelzon postulates for updates with concepts from the theory of logic-based abduction. A key feature of this new theory is that it presents a unified view of standard change operators and abductive operators rather than a new and independent change theory for abductive changes. Abductive operators reduce to standard change operators in some limiting cases.

Before we enter into specifics of the abductive change operators let us introduce the general idea behind the new operators.

2 Basic approach

Our general approach is to define abductive operators on top of regular change operators. To illustrate the idea, let \circ be a revision operator (in the sense of Gärdenfors), Σ an abductive domain theory, K a knowledge base and α a formula, all of them in a finite language. An abductive revision of K with α with respect to Σ requires finding an explanation γ for α based on Σ such that α will be a consequence of K revised with γ . Formally, an explanation for α (with respect to Σ according to K and \circ) is any formula γ such that $K \circ (\Sigma \wedge \gamma) \vdash \alpha$.⁴

There is a set of basic (or preferred) atoms Ab (called *abducibles*) that will be used to define explanations. Explanations are formulas built using atoms from Ab. Given α , we would like to find a formula which is an explanation for α . Let

 $AbEx(\alpha, K, \circ) = \{\gamma \text{ is an abducible formula consistent with } \Sigma : K \circ (\Sigma \land \gamma) \vdash \alpha \}$

A new operator \circ_a can be defined by letting $K \circ_a \alpha = K \circ (\Sigma \wedge \gamma)$, where $\gamma \in AbEx(\alpha, K, \circ)$. To make this idea precise, let $F : Form \to Form$ be a function such that if $AbEx(\alpha, K, \circ) \neq \emptyset$ then $F(\alpha) \in AbEx(\alpha, K, \circ)$, otherwise $F(\alpha) = \bot$. Define the abductive revision operator \circ_a by

$$K \circ_a \alpha = K \circ (\Sigma \wedge F(\alpha))$$

Notice that if γ is consistent with Σ and $\Sigma \wedge \gamma \vdash \alpha$ (i.e. γ is an explanation in the standard sense of logic-based abduction) then we have $\gamma \in AbEx(\alpha, K, \circ)$ (since $K \circ \beta \vdash \beta$). In this form, our definition extends the standard definition of abductive explanation.

We will work with a finite propositional language where K is the set of consequences of a formula Φ . In this case a revision operator is a function \circ that maps Φ and α into a formula $\Phi \circ \alpha$. We will denote by $F_c(\alpha)$ the cautious explanation of α , i.e., the disjunction of all abductive explanations of α . This function is well-defined since we are assuming that our language is finite.

It turns out that \circ_a will satisfy postulates similar to the AGM postulates if and only if F is essentially equal to F_c . Thus, our official definition of \circ_a will be $K \circ_a \alpha = K \circ (\Sigma \wedge F_c(\alpha))$. We will show that there is a selection involved in the definition of \circ_a , which is based on an order among all abductive explanations of α .

In the more general setting of an abstract change operator, our results show that it is natural to say that a change operator * is *abductive* when $\Phi * \alpha \vdash F_c(\alpha)$, where $F_c(\alpha)$ is the cautious explanation of α (with respect to *).

⁴By $\Sigma \wedge \gamma$ we mean the conjunction of all formulas in Σ and γ . Observe that Σ is playing a role of an integrity constraint as modeled by Katsuno-Mendelzon (see [15]).

In the next section we will precisely define domain theories and introduce a formal description of an abductive framework. Then we will introduce the abductive change operators corresponding to expansions and revisions and state a representation theorem for revisions. Next, we define abductive contractions in terms of revision operators and prove some of their properties. Finally, we explore abductive updates and suggest a way of defining updates which may lead to a representation theorem. Some concluding remarks and directions of research are presented in the last section.

3 Abductive change operators

Gärdenfors and his colleagues have introduced three basic operators in theory change: Expansions, Revisions and Contractions, and gave a set of postulates that the operators should satisfy in order to be considered rational change operators. Katsuno and Mendelzon introduced Update operators in order to model a type of change operators which are common in database applications, but they are not covered by revisions, contractions or expansions.⁵ In this section we will present abductive operators associated with each one of the change operators mentioned before.

Our original motivation for studying abductive operators was the view update problem in databases. In that setting, Φ can be regarded as consisting of two parts: One that we would always like to keep unmodified (the views or intensional part) and another part where we can make modifications (the extensional part that typically consists of abductive formulas). It is clear that updating such a database will consist of two steps: First, given the new fact α to be inserted, find the explanations of α (with respect to the views). Second, update the database with one of the explanations for α (it is in this step that the actual update occurs). The domain theory that we made reference in the introduction corresponds to the views in databases. With this idea in mind we introduce the notion of abductive framework.

Definition 3.1 A domain theory (cf. [16]) will be any consistent set of formulas and it will be usually denoted by Σ . A knowledge base Φ will be called *acceptable for* Σ if $\Phi \vdash \Sigma$, i.e., for every $\sigma \in \Sigma$, we have $\Phi \vdash \sigma$. Given a set of atoms Ab, called *abducibles*, any formula built using only atoms in Ab is called an *abducible formula*. The set of abducible formulas is denoted by AbForm. An *abductive framework* is any pair (Σ, Ab) where Σ is a domain theory and Ab is a set of abducible atoms.

Notice that, except for the acceptability of Φ , there are no constraints on the structure of Σ or Φ and any letter in the language can be designated to be abducible. We even allow the situation where there is no domain theory, i.e., every member of the domain theory is a tautology. This is equivalent to say that the domain theory is the empty set. The decision of what can be considered an explanation or abducible and what part of the theory is the domain theory is problem dependent and will form part of the design of the KB.

The goal is to expand, revise or update any possible extension of Σ (i.e., any acceptable Φ) using abduction based on a fixed abductive framework (Σ, Ab) .

Throughout the paper, a *change operator* will denote any function * that maps a knowledge base Φ and a formula α into a knowledge base $\Phi * \alpha$. A change operator * is said to satisfy *reflexivity* if for every α , $\Phi * \alpha \vdash \alpha$, and it satisfies *reciprocity* if $\Phi * \alpha \vdash \beta$ and $\Phi * \beta \vdash \alpha$ imply $\Phi * \alpha \equiv \Phi * \beta$. We will say that * satisfies the *Or rule* if for every abducible formula γ_1 and γ_2 : $\Phi * (\gamma_1 \lor \gamma_2) \vdash \Phi * \gamma_1 \lor \Phi * \gamma_2$.

⁵These operators were partly motivated by Winslett's possible world [22] approach for updating logical databases.

Definition 3.2 Let (Σ, Ab) be an abductive framework, Φ an acceptable knowledge base and * a change operator. The set of *abductive explanations* for α with respect to Φ and * is

 $AbEx(\alpha, \Phi, *) = \{\gamma \text{ is an abducible formula consistent with } \Sigma : \Phi * (\Sigma \land \gamma) \vdash \alpha \}.$

Remarks: (1) When the change operator * is understood from the context we will write $AbEx(\alpha, \Phi)$.

(2) We will show that expansion, revision and update operators satisfy the Or rule, therefore the disjunction of two explanations of a formula α is also an explanation of α , in other words, when * is any of those operators, $AbEx(\alpha, \Phi, *)$ is closed under \vee . In general this is not true for the conjunction of two explanations (for instance, there can be two explanations of α that are inconsistent with each other).

(3) The standard definition of an explanation of a formula α in an abductive framework (Σ, Ab) is any abducible formula γ such that $\Sigma \wedge \gamma$ is consistent and $\Sigma \wedge \gamma \vdash \alpha$. The change operators we are considering are such that $\Phi * \alpha \vdash \alpha$, hence if γ is an explanation in the standard sense then clearly $\gamma \in AbEx(\alpha, \Phi, *)$.

Following the ideas presented in the introduction, a selection function for Φ with respect to * will be any function $F_{\Phi}: Form \to Form$ such that

(i) For every formula α , $F_{\Phi}(\alpha)$ is an abductive explanation of α or, in case such explanation does not exist, $F_{\Phi}(\alpha) = \perp$.

(ii) If γ is an abducible formula then $F_{\Phi}(\gamma) \equiv \gamma$, i.e. the explanation for an abducible formula is the formula itself.

When there is no confusion about Φ we will write F instead of F_{Φ} . In this paper the most important selection function is the following:

Definition 3.3 Let (Σ, Ab) be an abductive framework, Φ an acceptable knowledge base, and *a change operator that satisfies the Or rule. If $AbEx(\alpha, \Phi, *) \neq \emptyset$, the *cautious explanation* of α with respect to Φ and * is the disjunction of all abductive explanations of α . We define the *cautious selection function* F_c by letting $F_c(\alpha)$ be equal to the cautious explanation of α . If $AbEx(\alpha, \Phi, *) = \emptyset$ then $F_c(\alpha) = \bot$.

Remarks: (1) Strictly speaking, F_c depends on Φ and *, so we should denoted it by $F_{\Phi,*}^c$, but to simplify the notation we drop Φ and * which should be clear from the context.

(2) Since we are working with a finite language, the set of all abductive explanations of α is finite modulo equivalent formulas, hence F_c is well defined.

Definition 3.4 Let (Σ, Ab) be an abductive framework and * be a change operator that satisfies the Or rule. The abductive operator associated with * is defined by

$$\Phi *_a \alpha = \Phi * (\Sigma \wedge F_c(\alpha))$$

where Φ is an acceptable knowledge base.

Now we show that F_c is, in some sense, the only selection function to be used. Let * be a change operator that satisfies reflexivity, reciprocity and the Or rule. Let F be any selection function for Φ with respect to *, and let F_c be the cautious selection function. Let $\Phi *^a \alpha = \Phi * (\Sigma \wedge F(\alpha))$. We will show that if $*^a$ satisfies reciprocity then $*^a = *_a$. First, since for every abducible formula γ , $\gamma \vdash F(\gamma)$, it follows from the reflexivity and the reciprocity of * that $\Phi *^a \gamma \equiv \Phi * (\Sigma \wedge \gamma)$. Second, since $F(\alpha) \vdash F_c(\alpha)$, we clearly have $\Phi *^a \alpha \vdash F_c(\alpha)$. Now, since $\Phi *^a F_c(\alpha) \equiv \Phi * (\Sigma \wedge F_c(\alpha))$, then $\Phi *^a F_c(\alpha) \vdash \alpha$. Therefore, by the reciprocity of $*^a$ we have $\Phi *^a \alpha \equiv \Phi *^a F_c(\alpha)$, which implies that $\Phi *^a \alpha \equiv \Phi * (\Sigma \wedge F_c(\alpha))$, that is to say $*^a = *_a$.

We are essentially saying that the only selection function that will induce a change operator satisfying Reciprocity is the cautious selection function.⁶ Hence, the only explanation we can be sure will always be part of $\Phi * \alpha$ is the cautious explanation of α . However, in most cases more specific explanations may appear in $\Phi * \alpha$. These explanations appear because there is another selection of preferred explanations involved in the construction of $\Phi * \alpha$, but this selection is implicitly built in the operator *. For the particular case of abductive revision operators, we will study the selection process with some care later in the paper (see 3.20). We would also like to remark that if condition (ii) above in the definition of a selection function F_{Φ} is dropped it is possible to find other selection functions, besides the cautious selection function, that will define change operators having the same properties as the one we have defined, however, must of the abductive frameworks or change operators with some form of abduction (like view updates in databases) that can be found in the literature assume condition (ii) [12, 13, 16, 11]. How interesting would it be to have functions without condition (ii) is a topic of future research. Some remarks regarding expansion operators defined by more specific selection functions can be found at the end of Section 3.3.

In the following sections we will study properties of several abductive change operators.

3.1 Abductive expansion

Expansion is the simplest form of change that one can perform on a knowledge base: Merely add the formula that is being incorporated. The expansion of a knowledge bases Φ with a formula α is denoted by $\Phi + \alpha$ and it is equal to $\Phi \wedge \alpha$. Expansion is axiomatically characterized by the following postulates ([7]):

$$(\mathbf{K}^+ \mathbf{1}) \ \Phi + \alpha \vdash \Phi \land \alpha$$

(**K**⁺**2**) If $\Phi \vdash \alpha$ then $\Phi + \alpha \equiv \Phi$.

(**K**⁺**3**) If $\Phi \vdash \Psi$ then $\Phi + \alpha \vdash \Psi + \alpha$.

Recall the definition of the abductive selection function given in 3.3 for the particular case of the operator + and also the definition of the abductive operator associated with + given in 3.4. Abductive expansion is then defined as follows:

Definition 3.5 Let (Σ, Ab) be an abductive framework and Φ an acceptable knowledge base. The *abductive expansion* of Φ with α , denoted by $\Phi +_a \alpha$, is defined by

$$\Phi +_a \alpha = \Phi \wedge F_c(\alpha)$$

Remarks: (1) The domain theory does not play an important role for the definition of $+_a$, for Φ acceptable, $\Phi \wedge F_c(\alpha) \equiv \Phi \wedge (\Sigma \wedge F_c(\alpha))$ (see Definition 3.4).

(2) Notice that when there is no abductive explanation for a formula α , then by definition $F_c(\alpha) \equiv \perp$, and hence $\Phi +_a \alpha \equiv \perp$.

(3) The definition of abductive expansion is very close to the standard framework for abduction (for instance, as presented in [12]). If we have to explain α in Φ we need to find a "minimal" formula that explains α . That is, a formula γ consistent with Φ such that γ together with Φ implies α .

⁶Reciprocity is an elementary feature of change operators. In fact, Revision, Contraction, Expansion and Update operators satisfy this property.

This minimal formula could be the disjunction of all possible consistent explanations of α . If there is no such explanation then the result of abduction will be the trivial knowledge base (usually, it is said that the explanation is not possible).

It is clear that $\mathbf{K}^+\mathbf{1}$ and $\mathbf{K}^+\mathbf{2}$ hold for $+_a$ (notice that when $\Phi \vdash \alpha$, then $F_c(\alpha)$ is a tautology). But $\mathbf{K}^+\mathbf{3}$ does not hold (as we will see below). However, the following partial version of $\mathbf{K}^+\mathbf{3}$ holds for $+_a$.

(AbK⁺3) If $\Phi \vdash \Psi$ and γ is an abducible formula then $\Phi + \gamma \vdash \Psi + \gamma$.

To see that $\mathbf{AbK^{+3}}$ holds just notice that if γ is an abducible formula, then $\Phi \wedge F_c(\gamma) \equiv \Phi \wedge \gamma$. One of the main consequences of $\mathbf{K^{+3}}$ is the following:

(**K**⁺**4**) If $\Phi + \alpha \vdash \beta$ then $\Phi + \alpha \vdash \Phi + \beta$.

To see that $\mathbf{K}^+\mathbf{4}$ still holds for $+_a$ even though $\mathbf{K}^+\mathbf{3}$ does not, notice that if $\Phi +_a \alpha \vdash \beta$ then $F_c(\alpha)$ is either \perp or an abductive explanation of β , and therefore $F_c(\alpha) \vdash F_c(\beta)$. Thus $\Phi \wedge F_c(\alpha) \vdash \Phi \wedge F_c(\beta)$.

The next theorem says that F_c is essentially the only selection function that can be used to create an abductive expansion operator that satisfies postulates $\mathbf{K}^+\mathbf{1}$, $\mathbf{K}^+\mathbf{2}$, $\mathbf{AbK}^+\mathbf{3}$ and $\mathbf{K}^+\mathbf{4}$ from the regular, non-abductive expansion operator.

Theorem 3.6 Let (Σ, Ab) be an abductive framework and Φ an acceptable knowledge base. Let $F = F_{\Phi}$ be a selection function for Φ and define an operator $+^*$ by $\Phi +^* \alpha = \Phi \wedge F(\alpha)$. Then $+^*$ satisfies $\mathbf{K}^+\mathbf{1}, \mathbf{K}^+\mathbf{2}, \mathbf{AbK}^+\mathbf{3}$ and $\mathbf{K}^+\mathbf{4}$ if and only if $\Phi \vdash F(\alpha) \leftrightarrow F_c(\alpha)$.

Proof: (\Leftarrow) It is the same argument as for F_c .

 (\Rightarrow) Observe first that any operator $+^*$ satisfying $\mathbf{K}^+\mathbf{1}$, $\mathbf{K}^+\mathbf{2}$ and $\mathbf{AbK}^+\mathbf{3}$ has the property that for every abducible formula γ , $\Phi +^* \gamma \equiv \Phi \land \gamma$. In fact, from $\mathbf{K}^+\mathbf{1}$ it suffices to show that $\Phi \land \gamma \vdash \Phi +^* \gamma$. Since $\Phi \land \gamma \vdash \Phi$, then from $\mathbf{AbK}^+\mathbf{3}$ we get that $(\Phi \land \gamma) +^* \gamma \vdash \Phi +^* \gamma$ and from $\mathbf{K}^+\mathbf{2}$ we have that $(\Phi \land \gamma) +^* \gamma \equiv \Phi \land \gamma$.

To complete the proof, from $\mathbf{K}^+\mathbf{4}$ it follows that whenever $\Phi \wedge \gamma \vdash \alpha, \ \Phi \wedge \gamma \vdash \Phi \wedge F(\alpha)$. Thus $\Phi \wedge F_c(\alpha) \vdash \Phi \wedge F(\alpha)$. But clearly $F(\alpha) \vdash F_c(\alpha)$, and therefore $\Phi \vdash F(\alpha) \leftrightarrow F_c(\alpha)$. \Box

To complete the picture, the next theorem shows that $+_a$ is uniquely characterized by the four postulates $\mathbf{K}^+\mathbf{1}$, $\mathbf{K}^+\mathbf{2}$, $\mathbf{AbK}^+\mathbf{3}$ and $\mathbf{K}^+\mathbf{4}$.

Theorem 3.7 Let (Σ, Ab) be an abductive framework. Let $+^*$ be an operator satisfying $\mathbf{K}^+\mathbf{1}$, $\mathbf{K}^+\mathbf{2}$, $\mathbf{AbK}^+\mathbf{3}$ and $\mathbf{K}^+\mathbf{4}$. Let $F = F_{\Phi}$ be a function such that for every formula α , $F(\alpha)$ is an abducible formula with $\Phi + F(\alpha) \vdash \alpha$. Assume that $\Phi + \alpha \vdash F(\alpha)$. Then $\Phi + \alpha \equiv \Phi \land F(\alpha)$. In fact F is a selection function for Φ and $\Phi \vdash F(\alpha) \leftrightarrow F_c(\alpha)$, i.e $\Phi + \alpha \equiv \Phi + \alpha$.

Proof: Since $\Phi + \alpha \vdash \Phi$, then from $\mathbf{AbK^+3}$ we get $(\Phi + \alpha) + F(\alpha) \vdash \Phi + F(\alpha)$. By the hypothesis, $\Phi + \alpha \vdash F(\alpha)$, thus from $\mathbf{AbK^+2}$ we have $\Phi + \alpha \vdash \Phi + F(\alpha)$. On the other hand, $\mathbf{K^+4}$ implies that $\Phi + F(\alpha) \vdash \Phi + \alpha$. Therefore, $\Phi + \alpha \equiv \Phi + F(\alpha)$.

By the same argument as in the proof of Theorem 3.6, it follows that for every abducible formula γ , $\Phi + \gamma \equiv \Phi \land \gamma$. Therefore $\Phi + \alpha \equiv \Phi \land F(\alpha)$. The rest of the proof follows from 3.6 \Box **Remark:** To prove the previous theorems we only need the weaker version of $\mathbf{K}^+\mathbf{4}$ which holds when α is an abducible formula.

Example 3.8 Consider the following modified version of the knowledge base Φ from the introduction:

$$\Phi = \begin{cases} r \to g \\ p \to g \\ g \to s \end{cases}$$

The propositional letter r can be read as *it rained last night*, the letter p as the sprinkler was on, g as the grass is wet and s as shoes are wet. Let $Ab = \{p, r\}$ and $\Sigma = \Phi$. Then we have that $\Phi +_a s = \Phi \land (r \lor p)$. Notice that s is consistent with Φ but $\Phi +_a s \neq \Phi \land s$. Also note that since $F_c(\neg s) = \bot$ then $\Phi +_a \neg s$ is the trivial knowledge base. \Box

The next theorem collects some other facts about $+_a$ that we will use later on.

Theorem 3.9 Let (Σ, Ab) be an abductive framework and Φ an acceptable knowledge base. Let α, β be formulas. Then

- (i) If γ is an abducible formula, then $\Phi +_a \gamma \equiv \Phi \wedge \gamma$
- (ii) $\Phi +_a \alpha \equiv \Phi +_a \beta$ if and only if $\Phi +_a \alpha \vdash \beta$ and $\Phi +_a \beta \vdash \alpha$.
- (iii) $(\Phi +_a \alpha) +_a \beta \equiv \Phi +_a (\alpha \land \beta).$

Proof: (i)was already proved. (ii) follows from $\mathbf{K}^+\mathbf{1}$ and $\mathbf{K}^+\mathbf{4}$. For (iii), from $\mathbf{K}^+\mathbf{1}$ we have that $(\Phi +_a \alpha) +_a \beta \vdash \Phi +_a (\alpha \land \beta)$ and the other direction follows easily.

The following example shows that the abductive expansion operator does not satisfy K^+3 .

Example 3.10 Let Φ be $(r \to g) \land (p \to g) \land \neg r$ and Ψ be $r \to g$. Let Σ be the empty domain theory and $Ab = \{p, r\}$. Then $\Phi +_a g = \Phi \land p$ and $\Psi +_a g = \Psi \land r$. Hence $\Phi +_a g \not\models \Psi +_a g$. \Box

Expansions (standard and abductive) are very conservative change operators. It is natural to ask what kind of expansion operators we could define if we allow explanations that are more "brave" than the cautious explanation. This situation is similar to the problem that motivates the definition of revision operators below. We will discuss more about abductive expansion operators after we introduce revisions in the next section.

Before we move to study abductive revisions let us observe that in the limiting case, when every atom is abducible $+_a$ and + are the same operator.

3.2 Abductive revision

Revision operators are defined in order to overcome the problem of trying to expand a knowledge base Φ with a formula that is inconsistent with Φ . We would like to accomplish that using abduction, so let (Σ, Ab) be an abductive framework and assume that we want to incorporate α into an acceptable knowledge base Φ where $\Phi +_a \alpha$ is the trivial base (there is no "easy" explanation for α). In the new knowledge base we want to have an explanation for α in terms of (Σ, Ab) , but at the same time we do not want to change Φ "too much". As indicated in the introduction, the underlying idea is to revise (in the usual way) Φ with an abducible formula γ such that the resulting theory is acceptable for Σ and implies α . Let us recall the definition of a revision operator. An operator \circ is called a revision operator if the operator satisfies the AGM postulates:

(R1) $\Phi \circ \alpha \vdash \alpha$.

(**R2**) If $\Phi \wedge \alpha$ is satisfiable, then $\Phi \circ \alpha \equiv \Phi \wedge \alpha$.

- (R3) If α if satisfiable, then $\Phi \circ \alpha$ is also satisfiable.
- (R4) If $\Phi_1 \equiv \Phi_2$ and $\alpha_1 \equiv \alpha_2$, then $\Phi_1 \circ \alpha_1 \equiv \Phi_2 \circ \alpha_2$.
- **(R5)** $(\Phi \circ \alpha) \land \beta \vdash \Phi \circ (\alpha \land \beta).$
- **(R6)** If $(\Phi \circ \alpha) \land \beta$ is satisfiable, then $\Phi \circ (\alpha \land \beta) \vdash (\Phi \circ \alpha) \land \beta$.

We will now introduce the new set of postulates that characterize abductive revisions. After the postulates are discussed we will present the typical schema to define abductive revision operators and some examples of abductive revisions that are defined on top of a well known (non-abductive) revision operator.

In order to define an abductive version of revision we will need the following notion.

Definition 3.11 Let (Σ, Ab) be an abductive framework and Φ an acceptable knowledge base. We will say that α is Ab-consistent with Φ if there is an abducible formula γ such that (i) $\Phi \wedge \gamma$ is consistent and (ii) $\Phi \wedge \gamma \vdash \alpha$.

Remarks: (1) α is Ab-consistent with Φ if and only if $\Phi +_a \alpha \neq \perp$.

(2) Ab-consistency is not reflexive, if α is Ab-consistent with Φ it is not necessarily the case that Φ is Ab-consistent with α .

For the propositional case (as presented by Katsuno-Mendelzon) there is no difference between knowledge bases and the knowledge to be inserted: both are propositional formulas. However, in an abductive framework there is a special class of formulas, the formulas in the domain theory. We do not make insertions of formulas that will modify or become part of the domain theory. This motivates our use of upper case Greek letters for knowledge bases and lower case letters for formulas to be inserted.

Definition 3.12 Let (Σ, Ab) be an abductive framework and \circ_a a change operator. Let Φ be a knowledge base acceptable for Σ and F_c the cautious selection function for Φ with respect to \circ_a . We will say that \circ_a is an *abductive revision operator* if it satisfies the following postulates:

(A0) For every α , $\Phi \circ_a \alpha \vdash \Sigma$ (i.e., $\Phi \circ_a \alpha$ is acceptable for Σ).

- (A1) For every α , $\Phi \circ_a \alpha \vdash \alpha$.
- (A2) If α is Ab-consistent with Φ , then $\Phi \circ_a \alpha \equiv \Phi +_a \alpha$

(A3) If γ is an abducible formula consistent with Σ , then $\Phi \circ_a \gamma$ is consistent.

(A4) If $\Phi_1 \equiv \Phi_2$ and $\alpha_1 \equiv \alpha_2$, then $\Phi_1 \circ_a \alpha_1 \equiv \Phi_2 \circ_a \alpha_2$.

- (A5) $(\Phi \circ_a \alpha) +_a \beta \vdash \Phi \circ_a (\alpha \land \beta).$
- **(A6)** If β is *Ab*-consistent with $\Phi \circ_a \alpha$, then $\Phi \circ_a (\alpha \wedge \beta) \vdash (\Phi \circ_a \alpha) +_a \beta$.
- (AA) $\Phi \circ_a \alpha \vdash F_c(\alpha)$.

Remarks: (1) For the limiting case when the set Ab of abducibles consists of every atom and Σ is the empty domain theory, then axioms **A1-A6** transform into **R1-R6** (recall that from 3.9(vi) $+_a$ becomes \wedge). So, the new axioms can be justified in the same way as the AGM postulates, i.e., essentially, they capture that the changes to Φ have to be minimal.

(2) Axiom A0 says that after changing (abductively) Φ we still have a knowledge base that implies the domain theory. In other words, Σ is playing the role of an integrity constraint. This is a basic fact about abduction: the domain theory is not supposed to change.

(3) Axiom **AA** (which we have called the *abductive axiom*) says that after inserting α the cautious explanation of α must also be true. This requirement says that \circ_a has an abductive nature. Since the cautious explanation can be considered the weakest of all explanation, **AA** imposes a *mild* condition over \circ_a .

(4) Observe that when γ is an abducible formula then from **A1** we have that $\gamma \vdash F_c(\gamma)$ and therefore $\Phi \circ_a \gamma \vdash F_c(\gamma)$, i.e., **AA** follows from the other postulates for abducible formulas.

(5) It will be shown that any operator \circ_a satisfying **A1-A6** will satisfy reciprocity. The operator will also have the property that if γ_1 and γ_2 are abducible formulas then $\Phi \circ_a (\gamma_1 \lor \gamma_2) \vdash \Phi \circ_a \gamma_1 \lor \Phi \circ_a \gamma_2$. As we have shown in the introduction, if F is a selection function for Φ and \circ_a such that $\Phi \circ_a \alpha \vdash F(\alpha)$ then $\Phi \circ_a F(\alpha) \equiv \Phi \circ_a F_c(\alpha)$.

Now we will present a basic schema to define abductive revision operators. This schema defines abductive revision operators in terms of regular revision operators as suggested in Section 2. In fact, after the representation theorem is proved, it will be clear that any abductive revision operator will be of this form. For the particular case of revision operators (recall Definition 3.4) we have:

Definition 3.13 Let (Σ, Ab) be an abductive framework and \circ a (regular) revision operator. For each acceptable knowledge base Φ , let F_c be the cautious selection function for Φ with respect to \circ . The (abductive) operator associated with \circ is $\Phi \circ_a \alpha = \Phi \circ (\Sigma \wedge F_c(\alpha))$.

Remarks: (1) Notice that from **R2** we obtain the following: Assume there is $\gamma \in AbEx(\alpha, \Phi, \circ)$ consistent with Φ (therefore $\Sigma \wedge \gamma$ is consistent with Φ) and let γ' be the disjunction of all such γ 's in $AbEx(\alpha, \Phi, \circ)$, then $\Phi \circ_a \alpha = \Phi \circ (\Sigma \wedge F_c(\alpha)) \equiv \Phi \wedge \gamma'$. Also, if $AbEx(\alpha, \Phi, \circ) = \emptyset$ then $\Phi \circ_a \alpha \equiv \bot$.

(2) Notice also that for every $\gamma \in AbEx(\alpha, \Phi, \circ)$ we have that $\gamma \vdash F_c(\alpha)$, but this does not necessarily imply that $\Phi \circ \gamma \vdash \Phi \circ F_c(\alpha)$. However, we will show later that there are many redundancies in $F_c(\alpha)$, in the sense that there is a subset of $AbEx(\alpha, \Phi, \circ)$ which suffices to define \circ_a .

(4) For every abducible formula γ consistent with Σ we have that $\Phi \circ_a \gamma = \Phi \circ (\Sigma \wedge F_c(\gamma)) \equiv \Phi \circ (\Sigma \wedge \gamma)$. This is because \circ satisfies reciprocity, i.e., if $\Phi \circ \alpha \vdash \beta$ and $\Phi \circ \beta \vdash \alpha$ then $\Phi \circ \alpha \equiv \Phi \circ \beta$. (see [7]).

The following theorem shows how standard revision operators relate to abductive operators.

Theorem 3.14 Let (Σ, Ab) be an abductive framework. An operator \circ_a satisfies axioms A0–A6 and AA if and only if there is a (regular) revision operator \circ^* such that for every acceptable Φ and every formula α we have $\Phi \circ_a \alpha = \Phi \circ_a^* \alpha$, where \circ_a^* is defined as in 3.13. In particular, for any abducible formula γ we have that $\Phi \circ_a \gamma \equiv \Phi \circ^* \gamma$.

Let us see an example.

Example 3.15 Consider the knowledge base in Example 3.8. Let $Ab = \{r, p\}$ and let Σ be the empty domain theory. We have that $F_c(s) = F_c(g) = p \lor r$ and $\Phi \circ_a s = \Phi \land (p \lor r)$. Also $AbEx(\neg s, \Phi, \circ) = AbEx(\neg g, \Phi, \circ) = \emptyset$, so $F_c(\neg g) = F_c(\neg s) = \bot$ and hence, $\Phi \circ_a \neg s = \Phi \circ_a \neg g$ is the

trivial knowledge base. Note that in this example we do not need to know what underlying revision operator \circ is used to define \circ_a since $F_c(s)$ is consistent with Φ and by **R2**, $\Phi \circ F_c(s) \equiv \Phi \wedge F_c(s)$.

Example 3.16 Consider the following knowledge base:

$$\Phi = \begin{cases} r \to g \\ p \to g \\ g \to s \\ \neg r \\ \neg p \end{cases}$$

Where s, g, r and p can be given the same interpretation as in Example 3.8 and $Ab = \{r, p\}$, $\Sigma = \emptyset$ are also as in Example 3.8. In this case we have that $Mod(\Phi) = \{\emptyset, \{s\}, \{s, g\}\}$ (we are working with Herbrand models).

Let \circ be Dalal's revision operator (see [3]). This operator is defined using a distance between a model M and a knowledge base Φ as follows: $dist(M, \Phi) = Min\{|M \triangle N| : N \models \Phi\}$, where \triangle is the symmetric difference. Define now a pre-order by: $N \leq_{\Phi} N'$ if and only if $dist(N, \Phi) \leq$ $dist(N', \Phi)$. Then $N \models \Phi \circ \alpha$ if an only if N is \leq_{Φ} -minimal among the models of α . Then $Mod(\Phi \circ r) = \{\{r\}, \{r, s\}, \{r, g, s\}\}, Mod(\Phi \circ p) = \{\{p\}, \{p, s\}, \{p, g, s\}\}, Mod(\Phi \circ (p \land r)) =$ $\{\{p, r\}, \{p, r, s\}, \{p, r, g, s\}\}$. Also, $\Phi \circ \neg r \equiv \Phi \circ \neg p \equiv \Phi \circ (\neg r \land \neg p) \equiv \Phi, \Phi \circ (\neg r \land p) \equiv \Phi \circ p$, $\Phi \circ (r \land \neg p) \equiv \Phi \circ r$. Then we have $AbEx(s, \Phi) = AbEx(\neg s, \Phi) = AbEx(g, \Phi) = AbEx(\neg g, \Phi) = \emptyset$. Thus the abductive revision of Φ with $s, \neg s, g$ and $\neg g$ is the trivial knowledge bases. However, with a different choice for the domain theory this problem is overcome. In fact, let $\Sigma = \{r \to g; p \to$ $g; g \to s\}$ and Ab as before. Then it is easy to verify that $\Phi \circ_a s \equiv \Sigma \land (p \lor r)$. The new choice of domain theory Σ can also be seen as a way of introducing views: In this example the formulas $\neg r$ and $\neg p$ in Φ are regarded as the extensional database and Σ as the views. However, we still have that $AbEx(\neg s, \Phi) = \emptyset$, this is due to the fact that the domain theory we are using is incomplete. We will come back to this problem later on. \Box

3.3 Representation theorems

In this section we will state a representation theorem for abductive revision operators in the same fashion as the representation theorems of Katsuno-Mendelzon ([15, 14]), Kraus, Lehmann and Magidor ([17]), Gärdenfors-Makinson ([9]) and Freund ([6]). Readers familiar with those papers will immediately realize the natural similarities between our proof and theirs.

Gärdenfors and Makinson [8] were the first to realize the connection that exists between the theory change and the theory of non-monotonic consequence relations. That connection has a significant impact on the proof of our results.⁷ In this section we will also present a more precise description of the abductive revision operators in terms of some orders of the abducible formulas.

Katsuno and Mendelzon gave a semantic characterization of revision operators based on orders over the collection of interpretations of the language of the knowledge base. They introduced the following notion of faithful assignments:

Definition 3.17 A faithful assignment is a map that assigns to each Φ a total pre-order \leq_{Φ} such that: (i) for every interpretation N and M, if M is a model of Φ then $M \leq_{\Phi} N$. (ii) If $M \in Mod(\Phi)$ and $N \notin Mod(\Phi)$ then $N \leq_{\Phi} M$ does not hold. (iii) If $\Phi \equiv \Psi$ then $\leq_{\Phi} = \leq_{\Psi}$.

⁷These similarities are even more apparent in [19], where we present a similar representation theorem for abductive non-monotonic consequence relations.

Models of the database can be taken as possible state of affairs in the world. The pre-order represents the preferences the agent may have regarding the plausibility of the different states of the world and it allows the selection of models for the revised knowledge base after a formula α is inserted. We will use the same approach to define a notion of explanation: given a faithful assignment, let $Expla(\alpha, \Phi, \leq_{\Phi})$ be the set of all abducible formulas γ consistent with Σ such that $Min(Mod(\Sigma \land \gamma), \leq_{\Phi}) \subseteq Mod(\alpha)$. Also we have the following selection function:

$$F_{\leq}(\alpha) = \begin{cases} \forall \{\gamma : \gamma \in Expla(\alpha, \Phi, \leq_{\Phi})\} & \text{If } Expla(\alpha, \Phi, \leq_{\Phi}) \neq \emptyset \\ \bot & \text{Otherwise} \end{cases}$$

With these concepts we are ready to state the representation theorem for abductive revision operators.

Theorem 3.18 Let (Σ, Ab) be an abductive framework. An operator \circ_a satisfies axioms A0-A6 and the abductive axiom AA if and only if there exists a faithful assignment that maps each acceptable Φ to a total pre-order \leq_{Φ} over the interpretations of the language such that:

$$Mod(\Phi \circ_a \alpha) = Min(Mod(\Sigma \wedge F_{\leq}(\alpha)), \leq_{\Phi}).$$

Proofs of most of the theorems can be found at the end of the paper. The following is a consequence of the previous theorem.

Corollary 3.19 Let Σ be any consistent set of formulas. Let \circ be a change operator that satisfies $\Phi \circ \alpha \vdash \Sigma$ for every α and every Φ acceptable for Σ . The operator \circ satisfies the AGM postulates if and only if there exists a faithful assignment that maps each acceptable Φ to a total pre-order \leq_{Φ} over the interpretations of the language such that:

$$Mod(\Phi \circ \alpha) = Min(Mod(\Sigma \wedge \alpha), \leq_{\Phi}).$$

Proof: Take Ab to be the set of all atoms, then use 3.18, 3.9 and Remark (1) after Definition 3.12. \Box

Notice that if Σ is the empty domain theory the corollary becomes the representation theorem of Katsuno-Mendelzon ([15], Theorem 3.3). Also, observe that here Σ is playing the role of an integrity constraint as modeled in [15].

We will now give a more precise description of \circ_a^* introduced in Theorem 3.14. We will show that \circ_a^* is implicitly defined by selecting some abductive explanations of α with respect to an order given by the original operator \circ^* . So let us fix a revision operator \circ^* .

We will need a method of comparing different explanations. One way to compare two explanations γ_1 and γ_2 of a formula α is by looking at the consequences of $\Phi \circ^* (\Sigma \wedge \gamma_i)$ which are also consequences of Φ and hence corroborate γ_i . Let us define $Corr(\gamma, \Phi) = \{\beta : \Phi \vdash \beta \& \Phi \circ^* (\Sigma \wedge \gamma) \vdash \beta\}$. Observe that $\beta \in Corr(\gamma, \Phi)$ if and only if $\Phi \lor \Phi \circ^* (\Sigma \wedge \gamma) \vdash \beta$. Hence we introduce the following pre-order:

$$\gamma \leq^{c}_{\Phi} \gamma' \quad \Leftrightarrow \quad \Phi \lor \Phi \circ^{*} (\Sigma \land \gamma') \vdash \Phi \lor \Phi \circ^{*} (\Sigma \land \gamma)$$

Notice that $\gamma \leq_{\Phi}^{c} \gamma'$ iff $Corr(\gamma, \Phi) \subseteq Corr(\gamma', \Phi)$, so when both γ and γ' are explanations of α, γ' is considered "better" than γ because there are more corroborating facts in Φ for γ' than for

 γ .⁸ It is clear that \leq_{Φ}^{c} is a reflexive and transitive relation but not antisymmetric. We define the strict relation $<^c_{\Phi}$ as usual and we say that $\gamma =^c_{\Phi} \gamma'$ if $\gamma \leq^c_{\Phi} \gamma'$ and $\gamma' \leq^c_{\Phi} \gamma$. We will say that γ is maximal with respect to a pre-order \leq if there is no γ' with $\gamma < \gamma'$. We define minimal elements analogously.

Let \leq_{Φ} be a faithful assignment (given by 3.19 with an empty domain theory) which maps each knowledge base Φ into a total pre-order of the interpretations of the language such that

$$Mod(\Phi \circ^* \alpha) = Min(Mod(\alpha), \leq_{\Phi}).$$

We define the following pre-order between formulas:

$$\gamma \leq^p_{\Phi} \gamma' \Leftrightarrow \Phi \circ^* (\Sigma \land \gamma) \vdash \Phi \circ^* (\Sigma \land (\gamma \lor \gamma'))$$

Notice that $\gamma \leq_{\Phi}^{p} \gamma'$ if and only if there are $N \in Mod(\Phi \circ^{*} (\Sigma \wedge \gamma))$ and $M \in Mod(\Phi \circ^{*} (\Sigma \wedge \gamma'))$ such that $N \leq_{\Phi} M$. Since \leq_{Φ} is total and the language is finite, we are assigning to each formula a natural number and $\gamma \leq_{\Phi}^{p} \gamma'$ amounts to saying that the same relation holds for the corresponding numbers. In particular this says that \leq^p_{Φ} is a total pre-order.⁹ We define $=^p_{\Phi}$ as usual.

The following theorem shows that the definition of \circ_a^* implicitly uses the orders \leq_{Φ}^p and \leq_{Φ}^c to select the "best" explanations. This is the selection that we referred to in the introduction.

Theorem 3.20 Let \circ^* be a (regular) revision operator and \circ^*_a be the abductive operator associated with \circ^* as defined in 3.13. Assume that $AbEx(\alpha, \Phi, \circ^*) \neq \emptyset$ and contains only formulas that are inconsistent with Φ . Let γ_i be the \leq^p_{Φ} -minimal elements of $AbEx(\alpha, \Phi, \circ^*)$, then $\Phi \circ^*_a \alpha \equiv$ $\Phi \circ^* (\Sigma \land (\gamma_1 \lor \cdots \lor \gamma_n)).$ Moreover, let $\{\delta_j : 1 \leq j \leq m\}$ be the \leq^c_{Φ} -maximal elements of $\{\gamma_i : 1 \leq i \leq n\}$, then $\Phi \circ^*_a \alpha \equiv \Phi \circ^* (\Sigma \land \delta_1) \lor \cdots \lor \Phi \circ^* (\Sigma \land \delta_m)$

We also have the following fact:

Fact 3.21 Under the same hypotheses of 3.20 let γ be a \leq_{Φ}^{c} -maximal element of $AbEx(\alpha, \Phi, \circ^*)$ and γ' be any abducible formula. If $\Phi \circ^* (\Sigma \wedge \gamma) \wedge \gamma'$ is consistent then $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \gamma'$

This fact together with Theorem 3.20 says that to define $\Phi \circ_a^* \alpha$ we select some \leq_{Φ}^c -maximal abducible explanations of α , i.e one of the formulas with one of the largest sets of corroborating facts. It also says that a \leq_{Φ}^{c} -maximal formula has the property that any two of its models can not be distinguished using abducible formulas.¹⁰ We can conclude that when an abducible formula γ (inconsistent with Φ) is a \leq_{Φ}^{c} -maximal explanation of some fact α , then for every abducible formula γ' , either $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \gamma'$ or $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \neg \gamma'$. This fact reaffirms that the selection of abducible formulas is a good selection, since each maximal formula does not leave explanations uncovered.

Example 3.22 Consider the following database (this is a minor modification of an example from [16])

$$\Phi = \left\{ \begin{array}{c} w, s \to p \\ p \to g \\ r \to g \\ r \to d \\ r \to \neg s \end{array} \right\} \Sigma$$

⁸Notice that $\Phi \lor \Phi \circ^* (\Sigma \land \gamma)$ is, by the Harper identity (see Section 3.4), the contraction of Φ with $\neg (\Sigma \land \gamma)$.

⁹If we reverse \leq_{Φ}^{p} , i.e., put $\alpha \leq \beta$ iff $\beta \leq_{\Phi}^{p} \alpha$, then \leq is a possibility order as in [5]. ¹⁰Notice that when every formula is abducible, a \leq_{Φ}^{c} -maximal formula is just the conjunction of literals that are true in a model.

The propositional letter r can be read as *it is raining*, the letter p as *sprinkler is on*, g as the grass is wet, s as *it is sunny day*, w as *it is a warm day* and d as the road is wet. Suppose we observe that the grass is wet. We will abductively revise Φ in order to incorporate g. Let Ab be $\{w, s, r\}$. Using Dalal's revision operator as \circ^* , we conclude that the \leq_{Φ}^c -maximal abductive explanations of g are ordered according to \leq_{Φ}^p as follows:

$$w \wedge s \wedge \neg r =_{\Phi}^{p} \neg w \wedge \neg s \wedge r \wedge <_{\Phi}^{p} w \wedge \neg s \wedge r$$

So, for Dalal's operator, $w \wedge \neg s \wedge r$ is too far away from the initial condition $\neg r \wedge \neg w$ in Φ , and therefore is not included as an explanation of g. Thus we have

$$\Phi \circ_a g \equiv \Phi \circ^* (\Sigma \wedge w \wedge s \wedge \neg r) \vee \Phi \circ^* (\Sigma \wedge \neg w \wedge \neg s \wedge r).$$

Consider now $\Phi' = \Phi \cup \{s\}$. We have $\Phi' \circ_a w \vdash g$. However, since simple abduction will only expand Σ minimally, the abduction of w will result in $\Sigma \wedge w$ but $\Sigma \wedge w \not\vDash g$. This shows that in some cases an explanation in our terms is not necessarily an explanation in the sense of standard abduction (but the converse is true as we have already shown).

For readers familiar with Gärdenfors' work [7] observe that for each \leq_{Φ}^{c} -maximal γ , $\Phi \circ^{*} \gamma$ corresponds to a theory revised using a maxi-choice function and the disjunction in 3.20 corresponds to a theory revised using a partial-meet contraction.

We conclude this section with some remarks regarding expansion operators. As we mentioned in Section 3.1, sometimes we would like to expand a KB with an explanation that is more specific than the cautious explanation. This can be done as follows. Let \leq_{Φ} be a total pre-order of $Mod(\Phi)$ (we do not require the order \leq_{Φ} to be faithful). Let define the set $Expla(\alpha, \Phi, \leq_{\Phi})$ and the cautious selection function F_{\leq} as before. The corresponding change operator (following 3.18) is

$$Mod(\Phi * \alpha) = Min(Mod(F_{<}(\alpha)), \leq_{\Phi}).$$

Notice that we do not need to include the models of Σ in the definition of $\Phi * \alpha$, i.e. we do not need $Mod(\Sigma \wedge F_{\leq}(\alpha))$ in the right hand side of the equation above, since $\Phi \vdash \Sigma$, and the order \leq_{Φ} is defined over the models of Φ . This operator * is an expansion operator in the sense that $\Phi * \alpha \vdash \Phi \land \alpha$ (i.e. $\mathbf{K}^+ \mathbf{1}$ holds). Of course, as we know from the results in Section 3.1 (see Theorem 3.7), the operator * cannot satisfy all properties $\mathbf{K}^+ \mathbf{2}$, $\mathbf{AbK}^+ \mathbf{3}$ and $\mathbf{K}^+ \mathbf{4}$ (unless \leq_{Φ} is the trivial pre-order where every two models are comparable). It is easy to verify that $\mathbf{K}^+\mathbf{4}$ holds. The other two axioms do not necessarily hold. In particular, even if α is a tautology we could have $\Phi * \alpha \neq \Phi$, which seems like a counterintuitive feature of *. This problem can be overcome by selecting a pre-order $<_{\Phi,\alpha}$ that depends on both the formula α to be incorporated and the KB Φ .¹¹ So, if we ask the order $<_{\Phi,\alpha}$ to be the trivial pre-order when α is in Φ , then we can show that $\mathbf{K}^+ \mathbf{2}$ holds. However, for this type of operators we do not necessarily have $\mathbf{K}^+ \mathbf{4}$. There is room for further developments. It seems that these expansion operators are closely related to the classic definition of abduction, but in this paper we have focused on the use of abduction to update KBs and we have not addressed the issue of using ideas of change theory to understand problems in the theory of abduction (see [2]). We believe that there are some connections to be explored. In particular we know that some forms of abductive reasoning can be formalized using ideas from theory change (see [19]).

¹¹Katsuno and Mendelzon already explored this possibility in the definition of revision operators [15].

3.4 Abductive contraction

Contractions are operations that retract incorrect beliefs from a knowledge base. Contraction and revision operators can both be defined in terms of each other by the so called Levi and Harper identities ([7]): Namely, let \circ be a revision operator. The *Harper identity* defines a contraction operator as follows:

$$\Phi \perp \alpha = \Phi \lor \Phi \circ \neg \alpha.$$

Conversely, the *Levy identity* says that if \perp is a contraction operator then a revision operator can be defined as follows: $\Phi \circ \alpha = (\Phi \perp \neg \alpha) + \alpha$.

We will define abductive contraction using the notion of abductive revision introduced in the previous section and the Harper identity.

Definition 3.23 Let (Σ, Ab) be an abductive framework, \circ a revision operator, Φ an acceptable knowledge base and α a formula. Let \circ_a be the operator defined as in 3.13 based on \circ . Then the *abductive contraction* of Φ with respect to α , denoted by $\Phi \perp_a \alpha$, is defined by

$$\Phi \perp_a \alpha = \Phi \lor \Phi \circ_a \neg \alpha$$

From the definitions it follows that if F_c is the cautious selection function for Φ with respect to \circ , then $\Phi \perp_a \alpha = \Phi \perp \neg F_c(\neg \alpha)$, where \perp is the contraction operator associated with \circ using the Harper identity. Clearly $\Phi \perp_a \alpha$ is acceptable for Σ . We have the following result:

Theorem 3.24 Let (Σ, Ab) be an abductive framework, \circ a (regular) revision operator and Φ an acceptable knowledge base. Let \perp_a be defined as in 3.23, then

(i) $\Phi \perp_a \alpha \not\vdash \alpha$.

- (ii) If $\neg \alpha$ is Ab-consistent with Φ then $\Phi \perp_a \alpha \equiv \Phi$.
- (iii) $(\Phi \perp_a \alpha) +_a \alpha \vdash \Phi$.
- (iv) $\Phi \vdash \Phi \perp_a \alpha$.

(v) If $\Phi_1 \equiv \Phi_2$ and $\alpha_1 \equiv \alpha_2$ then $\Phi_1 \perp_a \alpha_1 \equiv \Phi_2 \perp_a \alpha_2$.

- (vi) If α and β are abducible formulas then $\Phi \perp_a (\alpha \land \beta) \vdash \Phi \perp_a \alpha \lor \Phi \perp_a \beta$.
- (vii) If $\neg \alpha$ is Ab-consistent with $\Phi \perp_a (\alpha \land \beta)$ then $\Phi \perp_a \alpha \vdash \Phi \perp_a (\alpha \land \beta)$.

3.5 Abductive update

Katsuno and Mendelzon [14] introduced another type of change operator called Update which was motivated by Winslett's possible model approach to database updates [22]. Updates are used to make changes in a knowledge base to capture changes that occur in the world. Revisions try to correct misconceptions about the world represented in a knowledge base. In this section we present two directions on how abductive update operators may be defined. We recall the postulates for updates [14]:

(U1) $\Phi \diamond \alpha \vdash \alpha$.

(**U2**) If $\Phi \vdash \alpha$ then $\Phi \diamond \alpha \equiv \Phi$.

(U3) If Φ and α are satisfiable then $\Phi \diamond \alpha$ is also satisfiable.

(U4) If $\Phi_1 \equiv \Phi_2$ and $\alpha_1 \equiv \alpha_2$ then $\Phi_1 \diamond \alpha_1 \equiv \Phi_2 \diamond \alpha_2$.

(U5) $(\Phi \diamond \alpha) \land \beta \vdash \Phi \diamond (\alpha \land \beta).$

(U6) If $\Phi \diamond \alpha \vdash \beta$ and $\Phi \diamond \beta \vdash \alpha$ then $\Phi \diamond \alpha \equiv \Phi \diamond \beta$.

(U7) If Φ is complete then $(\Phi \diamond \alpha) \land (\Phi \diamond \beta) \vdash \Phi \diamond (\alpha \lor \beta)$.

 $(\mathbf{U8}) \ (\Phi_1 \lor \Phi_2) \diamond \alpha \equiv \Phi_1 \diamond \alpha \lor \Phi_2 \diamond \alpha$

One of the main differences between revision and update operators is given by the "disjunctive rule" **U8**. In an update, each world will receive equal and independent consideration. On the other hand, a revision is made in function of the knowledge base as a whole. So, for update operators, the basic operation is to change a single world. This fact must be reflected in the definition of the abductive update operator. Hence, we will use a different notion of abductive explanation which will depend on every single model of Φ .

To understand the approach better, we will first recall the semantic characterization of regular update operators.

Theorem 3.25 (Katsuno-Mendelzon [14]) An operator \diamond on a finite propositional language is an update operator (i.e. satisfies **U1-U8**) if and only if there exists a faithful assignment¹² which maps each interpretation M into a partial order \leq_M on the collection of all interpretations of the language such that

$$Mod(\Phi \diamond \alpha) = \bigcup_{M \in Mod(\Phi)} Min(Mod(\alpha), \leq_M).$$

Given an abductive framework (Σ, Ab) and an update operator \diamond , we define, similar to revision operators, the set of abductive explanations as follows:

$$AbEx(\alpha, \Phi, \diamond) = \{\gamma \in AbForm : \Sigma \land \gamma \text{ is consistent and } \Phi \diamond (\Sigma \land \gamma) \vdash \alpha \}.$$

Even though update operators work locally in each model, the previous definition is global over Φ , i.e., $\gamma \in AbEx(\alpha, \Phi, \diamond)$ whenever γ is an explanation for α in any model of Φ . A local version of this notion can be defined based on Katsuno and Mendelzon representation theorem for updates as follows:

Let \leq_M be a faithful assignment for \diamond (given by 3.25) and M an interpretation, then

$$AbEx(lpha, M, \leq_M) =$$

 $\{\gamma \in AbForm : \Sigma \land \gamma \text{ is consistent and } Min(Mod(\Sigma \land \gamma), \leq_M) \subseteq Mod(\alpha)\}$

This is equivalent to say, $\gamma \in AbEx(\alpha, M, \leq_M)$ if we have that α is true after updating M (using \leq_M) with γ . Clearly, for every $M \models \Phi$ it follows that $AbEx(\alpha, \Phi, \diamond) \subseteq AbEx(\alpha, M, \leq_M)$. But the converse is not necessarily true. It can be shown using results from [14] that $\Phi \diamond (\alpha \lor \beta) \vdash \Phi \diamond \alpha \lor \Phi \diamond \beta$. This implies that $AbEx(\alpha, \Phi, \diamond)$ and $AbEx(\alpha, M, \leq_M)$ are closed under \lor .

Those two different notions of explanation have naturally associated a cautious selection function and an abductive operator:

Definition 3.26 Let (Σ, Ab) be an abductive framework, \diamond an update operator and \leq_M the corresponding faithful assignment (given by theorem 3.25):

¹²By faithful assignment it is meant that the following conditions hold: For every interpretation $N, M \leq_M N$ and for all $N \neq M, N \leq_M M$. Notice the difference with the analogous notion used for revision operators.

(i) (Global function) Given an acceptable Φ , if $AbEx(\alpha, \Phi, \diamond) \neq \emptyset$ put $F_c(\alpha)$ to be the disjunction of all formulas in $AbEx(\alpha, \Phi, \diamond)$. Otherwise let $F_c(\alpha) = \bot$.

(ii) (Local function) Given a model $M \models \Sigma$, if $AbEx(\alpha, M, \leq_M) \neq \emptyset$ put $F_M(\alpha)$ to be the disjunction of all formulas in $AbEx(\alpha, M, \leq_M)$. Otherwise let $F_M(\alpha) = \bot$.

We have two operators associated with \diamond . Let $\Phi \equiv \bigvee_{i=1}^{n} \Phi_i$ with Φ_i a complete formula and M_i its unique model. Then let:

$$\Phi \diamond_a^g \alpha \stackrel{\text{def}}{=} \Phi \diamond F_c(\alpha) \quad \text{and} \quad \Phi \diamond_a^l \alpha \stackrel{\text{def}}{=} \bigvee_{i=1}^n \Phi_i \diamond F_{M_i}(\alpha)$$

The following example illustrates a situation where the appropriate operation required is an abductive update (this example is a minor modification of the one given by Katsuno and Mendelzon [14] to show the difference between update and revision).

Example 3.27 Consider the following scenario. There are two objects in a room: A book and telephone. The actions that a robot can perform in the room are two: To read the book and receive information through the telephone. To verify what kind of internal processing the robot is doing we can observe the robot. If it is holding the book the robot is reading. If it is holding the telephone it is receiving information. Let *b* represent "the robot is holding the book", *p* "the robot is holding the phone", *r* "the robot is reading the the book" and *w* "the robot is receiving information". Suppose also, that our original knowledge base consists of the following facts: Either the robot is reading the book or receiving information but not both.¹³ Let define Φ and Σ as follows:

$$\Phi = \left\{ \begin{array}{c} r \to b \\ w \to p \end{array} \right\} \Sigma$$
$$r \land \neg w \lor \neg r \land w$$

Suppose we would like to explain the state of the robot by doing observations. The set of abducibles, Ab, will be r and w. Suppose we observe that the robot is holding the book. We will compute $\Phi \diamond_a^g b$ and $\Phi \diamond_a^l b$ for the update operator \diamond defined by the faithful assignment \leq_M given by: $I \leq_M J$ iff the symmetric difference if I and M is a subset of the symmetric difference of J and M.

The models of Φ are $\{b, r\}$, $\{b, r, p\}$, $\{b, p, w\}$ and $\{p, w\}$. It is easy to check that $AbEx(b, \Phi, \diamond) = \{r, r \land \neg w, r \land w\}$: Since $Mod(\Phi \diamond (\Sigma \land r))$ are $\{b, r\}$, $\{b, p, r, w\}$, $Mod(\Phi \diamond (\Sigma \land r \land \neg w))$ are $\{b, r\}$, $\{b, r, p\}$ and $Mod(\Phi \diamond (\Sigma \land r \land w))$ is $\{b, p, r, w\}$. Observe that the disjunction of all explanations of b is equivalent to r, hence $F_c(b) = r$. Thus

$$\Phi \diamond_a^g b = \Phi \diamond (\Sigma \wedge r) = \Sigma \wedge r.$$

Therefore we have

$$Model(\Phi \diamond_{a}^{g} b) = \{\{b, r\}, \{b, r, p\}, \{b, p, r, w\}\}$$

Since the first three models of Φ are model of b then the corresponding local selection function is equivalent to a tautology (but notice that this is not a global explanation). For $M = \{p, w\}$, it is easy to check that $F_M(b) \equiv r \wedge w$. Thus we have that

$$Model(\Phi \diamond_a^l r) = \{\{b, r\}, \{b, r, p\}, \{b, p, w\}, \{b, p, r, w\}\}$$

¹³Although the robot can do parallel processing.

However, if we solve this problem using Dalal's revision operator (abductively) we only get the models $\{b, r\}$ and $\{b, r, p\}$, which are counterintuitive, since there is no reason to believe that the robot is not receiving information.

It is not difficult to show that **U1**, **U2**, **U4** and **U6** hold for both versions of abductive updates. We can also show that \diamond_a^g does not necessarily satisfy the disjunctive rule, but \diamond_a^l does. With respect to the other axioms we can say something if we interpret them as we did for the AGM axioms, i.e., we substitute \wedge by $+_a$ and consistency by Ab-consistency. The abductive version of **U5** and **U7** does not necessarily hold, even when Φ is complete (so there is no difference between \diamond_a^g and \diamond_a^l). For **U5** there is a remedy, it is enough to restrict the type of faithful assignment used. However, it is still an open question whether or not there is a similar representation theorem for abductive updates.

4 Final remarks and future work

In this article we have defined a new class of change operators based on abduction which have well known non-abductive change operators as limiting cases. Thus, the new theory presents a unified formalism for the study of both abductive and standard change operators. The idea underlying this new class of change operators is to carry out expansions in KBs by means of abduction as the basic "reasoning mechanism". Our results show that for our operators to be "rational" and abductive according to our definitions,¹⁴ the cautious explanation is the only explanation to be used during the operation. However, this uniqueness of explanation is deceptive since we have also shown that there are some choices that can be made during the definition of an abductive revision. The choices are based on an order among the abducible formulas (and the order is a reversed possibility ordering as presented in [5]) similar to the orders used to define non-abductive revisions. This similarity is reflected in our representation theorem. The representation theorem for abductive revision operators presented here is a generalization of the result in Katsuno-Mendelzon [15] for non-abductive revisions. Moreover, the domain theory is playing the role of an integrity constraint as modeled in [15]. A similar approach could be used to study other operators such as the one in [21] in terms of abduction by trying to define new abductive change operators based on regular operators.

The operators so defined do not preserve consistency (Φ and α can be consistent but $\Phi \circ_a \alpha$ can be inconsistent). There are two sources (not necessarily independent) for the inconsistency. One occurs when α is inconsistent with Σ . The second occurs when there is no abducible explanation even though $\Sigma \wedge \alpha$ is consistent.

This first source of inconsistency can be understood by looking at one of the limiting cases where every atom is considered abducible. In that case, the operators we define are just regular revision operators with the domain theory acting as a set of integrity constraints (see corollary 3.19). For the general case of an arbitrary abductive framework (Σ, Ab) we can view the loss of consistency as the fact that the set of integrity constraints is being violated. In this sense, abductive revision operators are very conservative since Σ is considered core knowledge that is not subject to revision and can not be changed. However, it is precisely the restrictions we have imposed on these operators not to modify Σ that have allowed us to model revision processes such as the one illustrated in the introduction and this can not be achieved with standard revision operators.

The second source of inconsistency can be addressed by adding escape explanations for each literal in the language. The concept of escape explanations was introduced by Konolige in [16]

¹⁴At least for expansion and revision operators.

to avoid inconsistencies in his mechanism of generating explanations in a logic-based abduction system. A similar approach has been also used in abductive logic programming [13]. How to introduce escape explanations in our framework of change operators is still an open question.

We would like to say a few words about the computational complexity of these new operators. Since we have shown that the operators (expansions, revisions, contractions and updates) can always be defined in terms of regular non-abductive operators, a lower bound of their complexity is given by the complexity of the underlying non-abductive operator. How complex these operators are depends on the class of KBs and the way the KBs are represented (models or set of formulas). Some results regarding the complexity of doing updates can be found in [10]. To the complexity of the underlying operator we must add the complexity of finding explanations. Again, this complexity will probably depend on the class of theories we choose to work with. Finding explanations in Horn theories will be probably easier than in more general theories. Results on the complexity of computing explanation in logic-based abduction systems can be found in [4]. Details of the complexity of the abductive operators are open problems.

It is interesting to remark that our notion of explanation differs from the usual notion of explanation in abductive reasoning by incorporating parts of Φ not necessarily in Σ in the reasoning process (see example 3.22).¹⁵

We have not considered in this article another very interesting approach to the problem of belief revision, namely base revision. It will be interesting to find out how the abductive method used here will work if the description of the world is given by a set of formulas that is not necessarily logically closed (i.e. a base).

Gärdenfors and Makinson [9] have shown that a revision operator can be viewed as a consequence relation in the following way: Given a background theory Φ we say that α non-monotonically entails β (with respect to the background theory Φ) if $\Phi \circ \alpha \vdash \beta$. In this setting, $AbEx(\alpha, \Phi)$ is the collection of abducible formulas that non-monotonically implies α . It is natural to study the corresponding consequence relation: $\alpha \models_{\alpha} \beta$ if $\Phi \circ_{a} \alpha \vdash \beta$. This relation does not satisfy the extended set of postulates (as given by Gärdenfors and Makinson), because the Or rule fails. However, it satisfies all axioms for the system **CL** defined by Kraus, Lehmann and Magidor [17] (i.e., it is a cumulative system that satisfies the Loop rule). Results on the properties of \models_{α} were presented in [19].¹⁶. This study is also important since we can show how some forms of abductive reasoning can be formalized using the results from the theory change and make precise connection between abduction and nonmonotonic consequence operators similar to the ones described in [20], in contrast to the work presented here that takes ideas from abduction to extend the theory of change operators.

5 Proofs

In this section we will present most of the proofs. The main result is 3.18, a representation theorem for abductive revision operators. We will give a complete proof of this result, even though some of the lemmas used are well known in the literature ([15, 14, 17, 9, 6]). In this way we will be able to get as a corollary (see 3.19) the representation theorem of Katsuno-Mendelzon ([15] Theorem 3.3), but we do not claim that our proof is easier than theirs.

Proof of 3.18

¹⁵Here is where the connection to the theory of non-monotonic consequence relations gives a better insight on the process we are modeling. This connection is explored in [19].

¹⁶Many of the proofs for this work were inspired by results in [17] and [9]

(\Leftarrow) Assume \leq_{Φ} is a faithful assignment and \circ_a is an operator defined using equations (i) and (ii). Notice, that if (ii) applies, then $\Phi \circ_a \alpha = \Phi \circ_a F_c(\alpha)$.

First we observe the following: For every abducible formula γ

$$Min(Mod(\Sigma \land \gamma), \leq_{\Phi}) \subseteq Mod(\Sigma \land F_{<}(\gamma))$$

and

$$Min(Mod(\Sigma \wedge F_{<}(\gamma)), \leq_{\Phi}) \subseteq Mod(\Sigma \wedge \gamma)$$

which implies:

$$Min(Mod(\Sigma \land \gamma), \leq_{\Phi}) = Min(Mod(\Sigma \land F_{<}(\gamma)), \leq_{\Phi})$$

Also, this implies that $AbEx(\alpha, \Phi, \circ_a) = Expla(\alpha, \Phi, \leq_{\Phi})$. From this, it clearly follows that $F_c = F_{<}$, and therefore **AA** holds. It is straightforward to check that axioms **A0**, **A1**, **A3**, **A4** hold.

To prove the other axioms we will need the following fact:

Fact 5.1 For every γ and γ' in *AbForm* if $(\Phi \circ_a \gamma) \land \gamma'$ is consistent then $(\Phi \circ_a \gamma) \land \gamma' \equiv \Phi \circ_a (\gamma \land \gamma')$.

Proof: Let $N \models (\Phi \circ_a \gamma) \land \gamma'$, then $N \models \Sigma \land \gamma \land \gamma'$. If $N' \leq_{\Phi} N$ with $N' \models \Sigma \land \gamma \land \gamma'$, then $N \leq_{\Phi} N'$ as N is \leq_{Φ} -minimal, thus $N \models \Phi \circ_a (\gamma \land \gamma')$. For the converse, first note that by hypothesis there is $N' \models (\Phi \circ_a \gamma) \land \gamma'$, thus $\Sigma \land \gamma \land \gamma'$ is consistent. Let then $N \models \Phi \circ_a (\gamma \land \gamma')$, since \leq_{Φ} is total and $N' \models \Sigma \land \gamma \land \gamma'$ then necessarily we have an $N \leq_{\Phi} N'$. Since $N \models \Sigma \land \gamma$, then $N \models (\Phi \circ_a \gamma)$, and thus $N \models (\Phi \circ_a \gamma) \land \gamma'$.

(A2) Assume that α is Ab-consistent with Φ , we want to show that $\Phi \circ_a \alpha \equiv \Phi +_a \alpha$. Assume first that $\alpha \in AbForm$, since α is Ab-consistent with Φ then α is also consistent with Φ . Then, by faithfulness $Min(Mod(\Sigma \wedge \alpha), \leq_{\Phi}) = Mod(\Phi \wedge \alpha) \equiv Mod(\Phi +_a \alpha)$ from (i) in Theorem 3.9.

If $\alpha \notin AbForm$, first note that $F_c(\alpha)$ is consistent with Φ , therefore from the previous case we get $\Phi \circ_a F_c(\alpha) \equiv \Phi + F_c(\alpha) \equiv \Phi \wedge F_c(\alpha)$ and then $\Phi \circ_a \alpha \equiv \Phi + F_a(\alpha)$.

(A5) If β is not Ab-consistent with $\Phi \circ_a \alpha$ then there is nothing to show. If β is Ab-consistent with $\Phi \circ_a \alpha$ we will show that (A5) and (A6) hold together, i.e., $\Phi \circ_a (\alpha \wedge \beta) \equiv (\Phi \circ_a \alpha) +_a \beta$.

(A6) Assume that β is *Ab*-consistent with $\Phi \circ_a \alpha$. Then $AbEx(\alpha, \Phi, \circ_a) \neq \emptyset$ if and only if $AbEx(\alpha \land \beta, \Phi, \circ_a) \neq \emptyset$. We will show that $\Phi \circ_a (\alpha \land \beta) \equiv (\Phi \circ_a \alpha) +_a \beta$. Let $\gamma_1, \ldots, \gamma_n$ be all the abducible formulas such that $(\Phi \circ_a \alpha) \land \gamma_i$ is consistent and $(\Phi \circ_a \alpha) \land \gamma_i \vdash \beta$. It is easy to see that

$$(\Phi \circ_a \alpha) +_a \beta \equiv (\Phi \circ_a \alpha) \land (\gamma_1 \lor \cdots \lor \gamma_n)$$

Now, from 5.1 we have $(\Phi \circ_a F_c(\alpha)) \wedge \gamma_i \equiv \Phi \circ_a (F_c(\alpha) \wedge \gamma_i)$. Hence $F_c(\alpha) \wedge \gamma_i \vdash F_c(\alpha \wedge \beta)$, since $F_c(\alpha) \wedge \gamma_i$ belongs to AbForm and $(\Phi \circ_a F_c(\alpha)) \wedge \gamma_i \vdash \alpha \wedge \beta)$. Therefore $(\Phi \circ_a F_c(\alpha)) \wedge F_c(\alpha \wedge \beta)$ is consistent and clearly $F_c(\alpha \wedge \beta) \vdash F_c(\alpha)$, hence $(\Phi \circ_a F_c(\alpha)) \wedge F_c(\alpha \wedge \beta) \equiv (\Phi \circ_a F_c(\alpha) \wedge F_c(\alpha \wedge \beta)) \equiv \Phi \circ_a F_c(\alpha \wedge \beta) \vdash \beta$. Thus for some $i, F_c(\alpha \wedge \beta) = \gamma_i$ and hence $F_c(\alpha \wedge \beta) \equiv F_c(\alpha) \wedge (\gamma_1 \vee \cdots \vee \gamma_n)$, i.e. $\Phi \circ_a F_c(\alpha \wedge \beta) \equiv (\Phi \circ_a F_c(\alpha)) \wedge (\gamma_1 \vee \cdots \vee \gamma_n)$. That is to say, $\Phi \circ_a (\alpha \wedge \beta) \equiv (\Phi \circ_a \alpha) + a \beta$. (\Rightarrow) Assume that the axioms **(A1)-(A6)** and the abductive axiom **AA** hold for the operator \circ_a . Let Φ be acceptable for Σ . We will define a total pre-order on the interpretations of the language.

Let Φ be acceptable for Σ . We will define a total pre-order on the interpretations of the language. We will say that an interpretation N is $normal^{17}$ if there is an abducible formula γ such that $N \models \Phi \circ_a \gamma$. First, we will define a relation over the normal interpretations and then we will extend it to all interpretations.

Let N_1 and N_2 be normal interpretations, we define $<_{\Phi}$, $=_{\Phi}$ and \leq_{Φ} as follows:

¹⁷In [18] a world N is called normal for a formula γ if (in ours terms) $N \models \Phi \circ_a \gamma$.

- $N_1 <_{\Phi} N_2$ if and only if $\forall \gamma_1, \gamma_2 \in AbForm$ such that $N_1 \models \Phi \circ_a \gamma_1$ and $N_2 \models \Phi \circ_a \gamma_2$ then $N_1 \models \Phi \circ_a (\gamma_1 \lor \gamma_2)$ and $N_2 \not\models \Phi \circ_a (\gamma_1 \lor \gamma_2)$.
- $N_1 =_{\Phi} N_2$ if and only if $N_1 \not\leq_{\Phi} N_2$ and $N_2 \not\leq_{\Phi} N_1$.
- $N_1 \leq_{\Phi} N_2$ if and only if $N_1 =_{\Phi} N_2$ or $N_1 <_{\Phi} N_2$.

We will show that \leq_{Φ} is a total pre-order over the normal interpretations and that the order is faithful. First, we need the following fact. Notice that part (a) says that every operator that satisfies A1-A6 will satisfy reciprocity.

Fact 5.2 (a) (*Reciprocity*) For every formulas α and β , $\Phi \circ_a \alpha \equiv \Phi \circ_a \beta$ if and only if $\Phi \circ_a \alpha \vdash \beta$ and $\Phi \circ_a \beta \vdash \alpha$.

(b) Let γ and γ' be abducible formulas then one of the following holds:

- (i) $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \circ_a \gamma$.
- (ii) $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \circ_a \gamma'$.
- (iii) $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \circ_a \gamma \lor \Phi \circ_a \gamma'$.

Proof: Observe that for abducible formulas, consistency and Ab-consistency are equivalent notions. (a) One direction follows directly from A1. For the other direction assume that $\Phi \circ_a \alpha \vdash \beta$. Then clearly β is Ab-consistent with $\Phi \circ_a \alpha$, thus by A5 and A6 we get that $\Phi \circ_a (\alpha \land \beta) \equiv (\Phi \circ_a \alpha) +_a \beta$. Hence by $\mathbf{K}^+ \mathbf{2}$ we get $\Phi \circ_a (\alpha \land \beta) \equiv \Phi \circ_a \alpha$. But analogously, $\Phi \circ_a (\alpha \land \beta) \equiv \Phi \circ_a \beta$.

(b) We consider three cases. Case 1: If $\Phi \circ_a (\gamma \lor \gamma') \vdash \neg \gamma'$, then from A1 we get $\Phi \circ_a (\gamma \lor \gamma') \vdash \gamma$. But clearly $\Phi \circ_a \gamma \vdash (\gamma \lor \gamma')$. Then by reciprocity $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \circ_a \gamma$. Case 2: If $\Phi \circ_a (\gamma \lor \gamma') \vdash \neg \gamma$ then by symmetry we get $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \circ_a \gamma'$. Case 3: Assume that $\Phi \circ_a (\gamma \lor \gamma') \nvDash \neg \gamma'$ and $\Phi \circ_a (\gamma \lor \gamma') \nvDash \neg \gamma$. Then γ and γ' are both consistent with $\Phi \circ_a (\gamma \lor \gamma')$. Thus, by A5 and A6 we have

$$\Phi \circ_a ((\gamma \lor \gamma') \land \gamma) \equiv \Phi \circ_a (\gamma \lor \gamma') \land \gamma$$

and also

$$\Phi \circ_a ((\gamma \lor \gamma') \land \gamma') \equiv \Phi \circ_a (\gamma \lor \gamma') \land \gamma'$$

and from A4

 $\Phi \circ_a ((\gamma \lor \gamma') \land \gamma) \equiv \Phi \circ_a \gamma$

and similarly

$$\Phi \circ_a ((\gamma \lor \gamma') \land \gamma') \equiv \Phi \circ_a \gamma'$$

Thus

$$\Phi \circ_a \gamma \lor \Phi \circ_a \gamma' \equiv \Phi \circ_a ((\gamma \lor \gamma') \land (\gamma \lor \gamma'))$$

and the result follows from A1.

Fact 5.3 The relation $<_{\Phi}$ is transitive.

Proof: Let $N_1 <_{\Phi} N_2$ and $N_2 <_{\Phi} N_3$ and assume $N_1 \not\leq_{\Phi} N_3$. Then there exist $\gamma_1, \gamma_3 \in AbForm$ such that $N_1 \models \Phi \circ_a \gamma_1$ and $N_3 \models \Phi \circ_a \gamma_3$ but either $N_1 \not\models \Phi \circ_a (\gamma_1 \lor \gamma_3)$ or $N_3 \models \Phi \circ_a (\gamma_1 \lor \gamma_3)$.

If $N_1 \not\models \Phi \circ_a (\gamma_1 \lor \gamma_3)$, then from 5.2 (b) we have that $\Phi \circ_a (\gamma_1 \lor \gamma_3) \equiv \Phi \circ_a \gamma_3$. Now, since N_2 is a normal model there exists $\gamma_2 \in AbForm$ such that $N_2 \models \Phi \circ_a \gamma_2$. Then, since $N_2 <_{\Phi} N_3$ and $N_3 \models \Phi \circ_a (\gamma_1 \lor \gamma_3)$ we have that $N_2 \models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$. And, again since $N_1 \models \Phi \circ_a \gamma_1$, and $N_1 <_{\Phi} N_2$ then it must be the case $N_2 \not\models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$, which is a contradiction.

If $N_3 \models \Phi \circ_a (\gamma_1 \lor \gamma_3)$, and $N_2 <_{\Phi} N_3$ and there exists $\gamma_2 \in AbForm$ such that $N_2 \models \Phi \circ_a \gamma_2$ then $N_2 \models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$. But since $N_1 <_{\Phi} N_2$ and $N_1 \models \Phi \circ_a \gamma_1$ then it must be the case that $N_2 \not\models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$ which is a contradiction. \Box

Fact 5.4 (i) Let N and M be normal models. Then $N <_{\Phi} M$ if and only if $\exists \gamma_N, \gamma_M$ abducible formulas such that $N \models \Phi \circ_a \gamma_N, M \models \Phi \circ_a \gamma_M$ and $N \models \Phi \circ_a (\gamma_N \lor \gamma_M)$ but $M \not\models \Phi \circ_a (\gamma_N \lor \gamma_M)$. (ii) In consequence, for N and M normal models, $N =_{\Phi} M$ if and only if for all abducible formulas γ and γ' such that $N \models \Phi \circ_a \gamma, M \models \Phi \circ_a \gamma'$ we have $N, M \models \Phi \circ_a (\gamma \lor \gamma')$.

Proof: (i) The if part comes directly form the definition of $\langle \Phi$. Assume that such γ_N and γ_M exist and let γ and γ' be any abducible formulas such that $N \models \Phi \circ_a \gamma$ and $M \models \Phi \circ_a \gamma'$. From 5.2(b) we get that $(\Phi \circ_a (\gamma \lor \gamma')) \land (\gamma_N \lor \gamma_M)$ and $(\Phi \circ_a (\gamma_N \lor \gamma_M)) \land (\gamma \lor \gamma')$ both are consistent. Hence using **A5**, **A6** and 3.9 we get that $(\Phi \circ_a (\gamma_N \lor \gamma_M)) \land (\gamma \lor \gamma') \equiv (\Phi \circ_a (\gamma \lor \gamma')) \land (\gamma_N \lor \gamma_M)$ and from this the result follows. (ii) follows from (i).

Fact 5.5 The relation $=_{\Phi}$ is an equivalence relation.

Proof: By definition the relation is reflexive and symmetric. The interesting case is when $N_1 \neq N_2$ and $N_2 \neq N_3$. Let $N_1 =_{\Phi} N_2$ and $N_2 =_{\Phi} N_3$. We will find γ'_1, γ'_3 in *AbForm* such that $N_1 \models \Phi \circ_a \gamma'_1$, $N_3 \models \Phi \circ_a \gamma'_3$ and $N_1, N_3 \models \Phi \circ_a (\gamma_1 \lor \gamma_3)$, then we will have that $N_1 \neq N_3$ and $N_3 \neq N_1$, and thus $N_1 =_{\Phi} N_3$. Let γ_1, γ_2 and γ_3 be *AbForm* such that $N_i \models \Phi \circ_a \gamma_i$, for i = 1, 2, 3 (These γ 's exist since N_i 's are normal models). Since $N_2 =_{\Phi} N_3$ then, from 5.4, we get $N_2, N_3 \models \Phi \circ_a (\gamma_2 \lor \gamma_3)$. Similarly, $N_1, N_2 \models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$ and $N_2, N_3 \models \Phi \circ_a (\gamma_1 \lor \gamma_2 \lor \gamma_3)$. Take $\gamma'_1 = \gamma'_3 = \gamma_1 \lor \gamma_2 \lor \gamma_3$. \Box

Now, we prove that \leq_{Φ} is a total pre-order. Reflexivity comes from the reflexivity of $=_{\Phi}$. For transitivity, let $N_1 \leq_{\Phi} N_2$ and $N_2 \leq_{\Phi} N_3$. We have four cases. (1) When $N_1 =_{\Phi} N_2$ and $N_2 =_{\Phi} N_3$, then $N_1 =_{\Phi} N_3$ follows from Fact 5.5. (2) When $N_1 =_{\Phi} N_2$ and $N_2 <_{\Phi} N_3$, the only case to consider is if $N_3 <_{\Phi} N_1$, but this case is impossible since by transitivity of $<_{\Phi}$ we have that $N_2 <_{\Phi} N_1$ contradicting the fact that $N_1 =_{\Phi} N_2$. (3) When $N_1 <_{\Phi} N_2$ and $N_2 =_{\Phi} N_3$, the situation is analogous to the second case. (4) When $N_1 <_{\Phi} N_2$ and $N_2 <_{\Phi} N_3$ follows from the transitivity of $<_{\Phi}$.

There are three conditions that need to be proved to show that the order is faithful: (i) for every pair of interpretations N and M, if M is a model of Φ then $M \leq_{\Phi} N$. (ii) If $M \in Mod(\Phi)$ and $N \notin Mod(\Phi)$ then $N \leq_{\Phi} M$ does not hold. (iii) If $\Phi \equiv \Psi$ then the relation \leq_{Φ} is the same for both Φ and Ψ . Condition (iii) follows from **A4**. Condition (i) is true since there is always $\gamma \in AbForm$ such that $M \models \gamma$, therefore for any $\gamma' \in AbForm$, $M \models \Phi \circ_a (\gamma \lor \gamma')$ (in particular M is normal) since $\Phi \circ_a (\gamma \lor \gamma') \equiv \Phi \land (\gamma \lor \gamma')$ from **A2** and (vi) in 3.9. Condition (ii) follows since, similar to condition (i), for any $\gamma, \gamma' \in AbForm$ if $M \models \Phi \circ_a \gamma$ then $\Phi \circ_a (\gamma \lor \gamma') = \Phi \land (\gamma \lor \gamma')$. Hence $N \not\models \Phi \circ_a (\gamma \lor \gamma')$ since $N \not\models \Phi$.

We extend \leq_{Φ} to an order on all interpretations as follows: If N is normal and M is not normal, then $N <_{\Phi} M$, and every two non-normal models are $=_{\Phi}$. It is clear that the extended relation \leq_{Φ} is a total pre-order and it is faithful.

Next, we show that for $\gamma \in AbForm$, $Mod(\Phi \circ_a \gamma) = Min(Mod(\Sigma \wedge \gamma), \leq_{\Phi})$. From A3, if γ is an abducible formula consistent with Σ then $\Phi \circ_a \gamma$ is consistent and hence there is a normal Nwith $N \models \gamma$. Let $Normal(\gamma)$ be the collection of all normal models of γ . We have just showed that for every abducible formula γ consistent with Σ , $Normal(\gamma)$ is not empty. Also, from A0 we have that $Normal(\gamma) = Normal(\Sigma \wedge \gamma)$. It follows from the way we extended \leq_{Φ} that for $\gamma \in AbForm$, $Min(Mod(\Sigma \wedge \gamma), \leq_{\Phi}) = Min(Normal(\gamma), \leq_{\Phi})$. We will show that for an abducible formula γ consistent with Σ , $Mod(\Phi \circ_a \gamma) = Min(Normal(\gamma), \leq_{\Phi})$.

 (\subseteq) By contradiction, assume that $M \in Mod(\Phi \circ_a \gamma)$ but $M \notin Min(Normal(\gamma), \leq_{\Phi})$. Then $\exists N <_{\Phi} M, N \in Min(Normal(\gamma), \leq_{\Phi})$. Then for any γ_1 such that $N \models \Phi \circ_a \gamma_1, N \models \Phi \circ_a (\gamma_1 \lor \gamma)$ and $M \not\models \Phi \circ_a (\gamma_1 \lor \gamma)$. Hence $\Phi \circ_a (\gamma_1 \lor \gamma) \land \gamma$ is consistent since N is a model, therefore γ is Ab-consistent

with $\Phi \circ_a (\gamma_1 \lor \gamma)$. Then from **A5** and **A6** we have $\Phi \circ_a (\gamma_1 \lor \gamma) \land \gamma \equiv \Phi \circ_a ((\gamma_1 \lor \gamma) \land \gamma) \equiv \Phi \circ_a \gamma$, contradicting that $M \not\models \Phi \circ_a (\gamma_1 \lor \gamma)$.

 $(\supseteq) \text{ Let } M \in Min(Normal(\gamma), \leq_{\Phi}), \text{ and let } \gamma' \in AbForm \text{ such that } M \models \Phi \circ_{a} \gamma' (\gamma' \text{ exists because } M \text{ is normal}). \text{ Since } Mod(\Phi \circ_{a} \gamma) \neq \emptyset \text{ and } Mod(\Phi \circ_{a} \gamma) \subseteq Min(Normal(\gamma), \leq_{\Phi}), \text{ then there is } N \text{ such that } N \models \Phi \circ_{a} \gamma \text{ and } N =_{\Phi} M \text{ (recall that } \leq_{\Phi} \text{ is total}). \text{ Then from 5.4, } M \models \Phi \circ_{a} (\gamma \vee \gamma') \text{ and hence } M \models (\Phi \circ_{a} (\gamma \vee \gamma')) \land \gamma. \text{ Therefore, } \gamma \text{ is Ab-consistent with } \Phi \circ_{a} (\gamma \vee \gamma') \text{ and thus from } A5, A6 \text{ and } 3.9 \text{ we get } (\Phi \circ_{a} (\gamma \vee \gamma')) \land \gamma \equiv \Phi \circ_{a} ((\gamma \vee \gamma') \land \gamma) \equiv \Phi \circ_{a} \gamma. \text{ Thus, } M \models \Phi \circ_{a} \gamma.$

Finally we show $Mod(\Phi \circ_a \alpha) = Min(Normal(F(\alpha)), \leq_{\Phi})$. Since \circ_a satisfies reciprocity, then from **AA** we get that $\Phi \circ_a \alpha \equiv \Phi \circ_a F_c(\alpha)$, and now the result follows, since we already have shown when α is an abducible formula. \Box **3.18**

Proof of 3.14

(\Rightarrow) Let \circ_a be an operator which satisfies axioms **A0-A6** and **AA** and let \leq_{Φ} be a faithful assignment given by 3.18. We can assume, with loss of generality, that \leq_{Φ} is defined for every Φ ¹⁸. Let \circ^* be the operator defined by the equation $Mod(\Phi \circ^* \alpha) = Min(Mod(\Sigma \wedge \alpha), \leq_{\Phi})$. Then clearly for every abducible formula γ we have that $\Phi \circ^* (\Sigma \wedge \gamma) = \Phi \circ_a \gamma$. Hence, it suffices to show that for every abducible formula γ we have that $\Phi \circ^*_a \gamma = \Phi \circ_a \gamma$. But this follows from the fact that $\Phi \circ^* (\Sigma \wedge \gamma) \equiv \Phi \circ^* (\Sigma \wedge F_c(\gamma))$ (see remark (4) after 3.13).

(\Leftarrow) By 3.18, it suffices to show that there is a faithful assignment \leq_{Φ} for every Φ acceptable for Σ . Apply 3.19 to \circ^* (with Ab the collection of all atoms and the trivial domain theory) and obtain a faithful assignment \leq_{Φ} such that

$$Mod(\Phi \circ^* \alpha) = Min(Mod(\alpha), \leq_{\Phi}).$$

From definition 3.13 of \circ_a^* and $F_c(\alpha)$ we get that if γ is an abducible formula then

$$Mod(\Phi \circ_a^* \gamma) = Min(Mod(\Sigma \wedge F(\gamma)), \leq_{\Phi}).$$

and from here we get that $AbEx(\alpha, \Phi, \circ^*) = Expla(\alpha, \Phi, \leq_{\Phi})$ (the last set defined as in 3.18). From this the result follows. \Box **3.14**

Proof of 3.20

First, we recall the following fact which follows from 5.2. Let \circ^* be a applied to an abductive framework where every atom is abducible and the domain theory is empty.

Fact 5.6 (a) (Reciprocity) For every formulas γ and γ' , $\Phi \circ^* \gamma \equiv \Phi \circ^* \gamma'$ if and only if $\Phi \circ^* \gamma \vdash \gamma'$ and $\Phi \circ^* \gamma' \vdash \gamma$.

(b) For every formulas γ and γ' one of the following holds :

- (i) $\Phi \circ^* (\gamma \lor \gamma') \equiv \Phi \circ^* \gamma$.
- (ii) $\Phi \circ^* (\gamma \lor \gamma') \equiv \Phi \circ^* \gamma'.$
- (iii) $\Phi \circ^* (\gamma \lor \gamma') \equiv \Phi \circ^* \gamma \lor \Phi \circ^* \gamma'$.

Fact 5.7 (i) If $\Phi \circ^* (\Sigma \land \gamma') \vdash \Phi \circ^* (\Sigma \land \gamma)$ then $\gamma \leq^c_{\Phi} \gamma'$.

(ii) If γ' is inconsistent with Φ then $\gamma \leq^{c}_{\Phi} \gamma'$ implies $\Phi \circ^{*} (\Sigma \wedge \gamma') \vdash \Phi \circ^{*} (\Sigma \wedge \gamma)$.

(iii) If γ is consistent with Φ then γ is a \leq_{Φ}^{c} -maximum.

¹⁸The only case not covered is when Φ is not acceptable, then let \leq_{Φ} be any total pre-order which satisfies faithfulness.

Proof: (i) Obvious. (ii) Assume that γ' is inconsistent with Φ and $\gamma \leq_{\Phi}^{c} \gamma'$. Let M be a model of $\Phi \circ^* (\Sigma \wedge \gamma')$. Then M is a model of $\Phi \vee \Phi \circ^* (\Sigma \wedge \gamma)$, and since $\Phi \circ^* (\Sigma \wedge \gamma)$ implies γ , then M is a model of $\Phi \circ^* (\Sigma \wedge \gamma)$. (iii) If γ is consistent with Φ then $\Phi \circ^* (\Sigma \wedge \gamma) = \Phi \wedge \gamma$. Thus $\Phi \lor \Phi \circ^* (\Sigma \land \gamma) \equiv \Phi.$

Fact 5.8 (i) If $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \Phi \circ^* (\Sigma \wedge \gamma')$ then $\gamma \leq^p_{\Phi} \gamma'$

(ii) Let γ_i be formulas and let γ_{i_k} be its \leq_{Φ}^p -minimal (there could be more than one since \leq_{Φ} is a pre-order). Then $\Phi \circ^* (\Sigma \land (\gamma_1 \lor \cdots \lor \gamma_n)) \equiv \Phi \circ^* c(\Sigma \land \gamma_{i_1}) \lor \cdots \lor \Phi \circ^* (\Sigma \land \gamma_{i_k}).$

Proof: From 5.6(b) we get the following:

(a) If $\gamma <_{\Phi}^{p} \gamma'$ then $\Phi \circ^{*} (\Sigma \wedge \gamma) \equiv \Phi \circ^{*} (\Sigma \wedge (\gamma \vee \gamma'))$.

(b) If γ and γ' are both \leq_{Φ}^{p} -minimal in $\{\gamma, \gamma'\}$ then $\Phi \circ^{*} (\Sigma \wedge (\gamma \vee \gamma')) \equiv \Phi \circ^{*} (\Sigma \wedge \gamma) \vee \Phi \circ^{*} (\Sigma \wedge \gamma')$. Now, (i) follows easily from (a) and (b), and (ii) follows from 5.6(b).

Fact 5.9 Let γ_i be the \leq_{Φ}^p -minimal elements of $AbEx(\alpha, \Phi, \circ^*)$ and put $\gamma^* = \gamma_1 \vee \cdots \vee \gamma_m$. Let $\{\delta_1,\ldots,\delta_n\} = Max(\{\gamma_i: 1 \le i \le m\}, \le^c_{\Phi}), \text{ and } \delta^* = \delta_1 \lor \cdots \lor \delta_n. \text{ Then } \Phi \circ^* (\Sigma \land \gamma^*) \equiv \Phi \circ^* (\Sigma \land \delta^*).$

Proof: Let $A = \{\gamma \in AbForm : \text{for some } i, \gamma_i \leq_{\Phi}^c \gamma \text{ and } \gamma \text{ is inconsistent with } \Phi\}$. From 5.7(ii) if $\gamma \in A$ then $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \Phi \circ^* (\Sigma \wedge \gamma_i)$ (some *i*), thus $\gamma \in AbExp(\alpha, \Phi, \circ^*)$ and also $\gamma \leq^p_{\Phi} \gamma_i$ (from 5.8(i)). Thus for some $j, \gamma = \gamma_j$. Therefore $Max(A, \leq_{\Phi}^c) = \{\delta_1, \dots, \delta_n\}$.

To prove the claim, first notice that $\Phi \circ^* (\Sigma \wedge \delta^*) \vdash \gamma^*$ thus from 5.6(a) it suffices to show that $\Phi \circ^* (\Sigma \wedge \gamma^*) \vdash \delta^*$. So, assume, towards a contradiction, that $\Phi \circ^* (\Sigma \wedge \gamma^*) \not\vdash \delta^*$ and let $\gamma' \equiv \gamma^* \wedge \neg \delta^*$. Then $\gamma' \in AbForm$. Since γ' is consistent with $\Phi \circ^* (\Sigma \wedge \gamma^*)$, from **R6** we have that $(\Phi \circ^* (\Sigma \wedge \gamma^*)) \wedge \gamma' \equiv \Phi \circ^* (\Sigma \wedge (\gamma^* \wedge \gamma')) \equiv \Phi \circ^* (\Sigma \wedge \gamma')$. Hence $\gamma^* \leq^c_{\Phi} \gamma'$ (from 5.7(i)). Therefore $\gamma' \in A$ (notice that from 5.8(ii) $\gamma^* \equiv \gamma_i$ for some *i* and clearly γ' is inconsistent with Φ) and hence there is j such that $\gamma' \leq_{\Phi}^{c} \delta_{j}$. But $\Phi \circ^{*} (\Sigma \wedge \gamma') \vdash \neg \delta_{i}$, a contradiction.

The proof of 3.20 now follows from 5.8 and 5.9.

Proof of 3.21

Since \circ^* is a revision operator $(\Phi \circ^* (\Sigma \land \gamma)) \land \gamma' \equiv \Phi \circ^* (\Sigma \land (\gamma \land \gamma'))$. Thus, $\gamma \land \gamma'$ is in $AbEx(\alpha, \Phi)$ and $\gamma \leq_{\Phi}^{c} \gamma \wedge \gamma'$. Therefore, by the maximality of γ , we have that $\gamma \wedge \gamma' \leq_{\Phi}^{c} \gamma$ and hence (using 5.7(ii)) $\Phi \circ^* (\Sigma \wedge \gamma) \vdash \gamma'$. $\Box 3.21$

Proof of 3.24

(o) follows from A0 for \circ_a . (i) follows from the fact that $\Phi \circ_a \alpha \vdash \alpha$. (ii) From A2 we know that $\Phi \circ_a \neg \alpha = \Phi +_a \neg \alpha$, and hence $\Phi \lor \Phi \circ_a \neg \alpha \equiv \Phi$. (iii) If $(\Phi \perp_a \alpha) +_a \alpha$ is inconsistent there is nothing to show, otherwise let γ be an abducible formula such that $(\Phi \perp_a \alpha) \wedge \gamma$ is consistent and $(\Phi \perp_a \alpha) \land \gamma \vdash \alpha$. Then γ is inconsistent with $\Phi \circ_a \neg \alpha$ and thus, $(\Phi \lor \Phi \circ_a \neg \alpha) \land \gamma \equiv \Phi \land \gamma$. Hence $(\Phi \perp_a \alpha) +_a \gamma \vdash \Phi$. (iv) follows from the definition of \perp_a . (v) follows from A4 for \circ_a . (vi) Since $\neg \alpha$ and $\neg \beta$ are abducible formulas then from 5.2 (b) $\Phi \circ_a (\neg \alpha \lor \neg \beta) \vdash \Phi \circ_a \neg \alpha \lor \Phi \circ_a \neg \beta$. Hence $\Phi \lor \Phi \circ_a (\neg \alpha \lor \neg \beta) \vdash (\Phi \lor \Phi \circ_a \neg \alpha) \lor (\Phi \lor \Phi \circ_a \neg \beta).$ (vii) There are two cases to consider: (a) If $\neg \alpha$ is Ab-consistent with Φ then $\Phi \circ_a \neg \alpha \equiv \Phi +_a \neg \alpha \vdash \Phi$. Therefore $\Phi \perp_a \alpha \vdash \Phi$ and hence $\Phi \perp_a \vdash (\alpha \land \beta)$. (b) If $\neg \alpha$ is not Ab-consistent with Φ , then $\neg \alpha$ is Ab-consistent with $\Phi \circ_a (\neg \alpha \lor \neg \beta)$. Then from A6 we get that $\Phi \circ_a \neg \alpha \equiv \Phi \circ_a ((\neg \alpha \lor \neg \beta) \land \neg \alpha) \vdash \Phi \circ_a (\neg \alpha \lor \neg \beta)$. Therefore $\Phi \perp_a \neg \alpha \vdash \Phi \perp_a (\alpha \lor \beta)$. $\Box 3.24$

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