Analytic k-spaces

S. Todorčević and C. Uzcátegui

October 4, 2002

Abstract

We study sequential convergence in spaces with analytic topologies avoiding thus a number of standard pathologies. For example, we identify bisequentiality of an analytic space as the Frechet property of its square. We show that a countable Frechet group is metrizable if and only if its topology is analytic. We also investigate the diagonal sequence properties and show their productiveness in the class of analytic spaces.

1 Introduction

This is a continuation of our paper [20] where we study effective versions of some standard topological problems and results. Thus we restrict ourselves to (regular) countable topological spaces X with the property that the family τ_X of all open subsets of X is analytic (i.e., continuous image of the irrationals) as a subset of the Cantor cube 2^X. We call any such X an *analytic space*. Many of the standard examples of countable spaces are analytic. For example, the *Arens space* [1], the *Arhangelski-Franklin space* [3], and the countable *Sequential fan* [12] are all analytic spaces. On the other hand, many topological applications to the study of, say, weak topology of Banach spaces require results about countable analytic spaces (see e.g., [2]).

Recall, that X is said to be a *k-space* if and only if an arbitrary subset of X is closed just in case its intersection with an arbitrary compact subset of X is closed (see, e.g., [15]). In our context this reduces to the more familiar class of *sequential spaces*. Recall that X is said to be sequential if for every non closed $A \subseteq X$ there is a sequence of elements of A converging to a point outside of A. If we require a sequence of elements of A to converge to an arbitrary point of the closure of A we get the considerably more restrictive class of *Frechet spaces*. It is usually in relation to these classes of spaces that one considers various ways to obtain a converging sequence out of a sequence of converging sequences. Recall that the *diagonal-sequence property* states that if $\{x_{nk}\}$ is a double-indexed sequence of members of X such that for some $x \in X$ and all n, $x_{nk} \to_k x$. Then for each n we can choose $k(n)$ such that $x_{nk(n)} \to_n x$. If we require that some infinite subsequence of $\{x_{nk(n)}\}$ converges to x rather than the sequence itself, we get the *weak diagonal sequence property.* Note that the diagonal sequence property and the weak diagonal sequence property are formally incomparable with the Frechet property. Consider, for example, Arens space and the Sequential fan. The former has the diagonal sequence property but it is not Frechet while the later is Frechet but it fails even the weak diagonal sequence property. It turns out that in the context of analytic spaces the diagonal sequence property is as restrictive as first countability (metrizability) (see [17] and [20]]). We give here a variation of this result by showing that an analytic sequential space with the diagonal sequence property is weakly first countable. Recall that we say that X is *weakly first countable* if for every $x \in X$ we can find a decreasing sequence of sets $B(x,n) \ni x$ such that a set V is open iff for all $x \in V$, there is m with $B(x,m) \subseteq V$. Note that the topology of an arbitrary countable weakly first countable space is analytic (in fact, $F_{\sigma\delta}$). For example, the Arens space is a typical example of a weakly first countable space. Note that every weakly first countable space has the diagonal sequence property and that every Frechet weakly first countable space is in fact first countable.

Recall now that X is said to be *bisequential* if for every ultrafilter $\mathcal{U}$ over X converging to some point x there is a sequence $A_n \in \mathcal{U}$ converging to x. Clearly every bisequential space is Frechet but not vice versa. Consider, for example, the Sequential fan. Note also that every bisequential space has the weak diagonal sequence property but not vice versa. Consider, for example, the Arens space. We show however that the two properties jointly characterize bisequentiality in the class of analytic space. Thus we show that every analytic Frechet space with the weak diagonal sequence property is bisequential. We give some application of this result to the study of products as well as to the study of countable topological groups. For example, we show that the square of an analytic Frechet space X is Frechet if and only if X contains no copy of the sequential fan $S(\omega)$. As another application we show that analytic Frechet groups are metrizable solving thus the effective version of the well known problem of Malyhin (see, e.g., [12]). The preservation of the weak diagonal sequence property in products of analytic spaces seems curiously related to the problem whether the Sequential fan is a test space for the failure of this property in the class of analytic spaces. This can be seen from the fact which we show here which says that the sequential fan does not embed into the product of two analytic spaces with the weak diagonal sequence property. This can be regarded as a proof of the effective version of a conjecture of Nogura[10]. The proof of the unrestricted version of Nogura's conjecture is given in [19] and our proof here can be regarded as its effective version.

2 Test spaces

In this section we introduce some critical examples of analytic spaces for testing convergence properties introduced above. The first such a space is the Arens space [1], denoted by S_2, the space on $\omega^{\leq 2}$ with the following topology. Each sequence of length 2 is isolated, a basic nbhd of the sequence $\langle n \rangle$ consists of all sequences of the form $\langle n, m \rangle$ for all but finitely many m's and, finally, a set U is a basic nbhd of the empty sequence $\emptyset$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ and an integer n with the property that $\langle m \rangle, \langle m, k \rangle \in U$ for all $m \geq n$ and all $k \geq f(m)$. The following well known fact gives a clear indication of the importance of the Arens space for this area (see, e.g., [14]).

Proposition 2.1 *A sequential space is Frechet if and only if it contains no homeomorphic copy of the Arens space S_2.*

The sequential fan, denoted by $S(\omega)$, is the space defined over $\mathbb{N}\times\mathbb{N}\cup\{\infty\}$ where all points in $\mathbb{N} \times \mathbb{N}$ are isolated and the nbhd filter of ∞ is generated by the sets of the form $U_f = \{(n, m) \in \mathbb{N} \times \mathbb{N} : \ m \geq f(n)\} \cup \{\infty\}$ for $f \in \mathbb{N}^{\mathbb{N}}$. The sequential fan $S(\omega)$ is a typical Frechet space without the weak diagonal sequence property. The following well known fact shows that inside the class of Frechet spaces $S(\omega)$ is a test space for the weak diagonal sequence property (see, e.g., [14]).

Proposition 2.2 *A Frechet space has the weak diagonal sequence property if and only if it contains no homeomorphic copy of $S(\omega)$.*

The following result of Nyikos[11] shows that inside the class of topological groups the two spaces perform the same task of testing

Proposition 2.3 *A topological group contains a copy of the Arens space if and only if it contains a copy of the sequential fan.*

Corollary 2.4 *A sequential topological group is Frechet if and only if it has the weak diagonal sequence property.*

Note that we are working inside the class of countable spaces X, so if such an X contains a copy of S_2 or $S(\omega)$ then it also contains closed subspaces homeomorphic to S_2 or $S(\omega)$. This will be implicitly used below. A natural generalization of the Arens space is the Arkhanglel'skiĩ-Franklin space S_ω [3]. It is defined in [3] as the direct limit of the sequence S_n ($n \in \omega$), where S_{n+1} is related to S_n the same way the Arens space S_2 is related to the converging sequence S_1. We shall, however, not work with this description of the space S_ω. We have a new parametrized class spaces which for a particular choice of the parameter gives a space homeomorphic to S_ω. We should note, however, that the space S_ω is another test space, especially when one considers high sequential orders. For example, to show that a given sequential space X has sequential order ω_1 one typically shows this by embedding S_ω in X (see [3], [6], [20]).

3 Selective points

A point $x \in X$ is called a *Frechet point*, if for every $A \subseteq X$ with $x \in \overline{A}$ there is a sequence $x_n \in A$ converging to x (in short, X is Frechet at x). Analogously, we define the notion of a *bisequential point*. We will say that x is a q$^+$-*point*, if for every A with $x \in \overline{A}$ and every partition $A = \bigcup_n F_n$ of A into finite sets, there is a subset B of A such that $x \in \overline{B}$ and $|B \cap F_n| \leq 1$ for all n. We say that x is a p$^+$-*point* if given any decreasing sequence P_n of subset of X such that $x \in \overline{P_n}$ for all n, there is a set P such that $x \in \overline{P}$ and $P \subseteq^* P_n$ (i.e. $P \setminus P_n$ is finite) for all n. We note that a point which is at the same time a Frechet point and a p$^+$- point is also called in the literature a *countably bisequential point*, or a *strongly Frechet point* [9, 14]. We will say that x is a *selective* point if it is both a p$^+$and q$^+$-point.

Proposition 3.1 *Every point of a sequential space is a q$^+$-point.*

Proof. Let A be a subset of X and F_m be a partition of A into finite pieces. Let $x \in \overline{A}$, we will show by induction on the sequential rank of x in A that there is $F \subseteq A$ such that $|F \cap F_m| \leq 1$ and $x \in \overline{F}$. Recall that $A^{(1)}$ is the set of all limits of convergent sequences in A. Let $A^{(0)} = A$, $A^{(\alpha+1)} = [A^{(\alpha)}]^{(1)}$ and $A^{(\beta)} = \cup_{\alpha<\beta} A^{(\alpha)}$ for β a limit ordinal. The *sequential closure* of A is the set $A^{(\omega_1)}$. It is clear that if $x \in A^{(1)}$ then the result follows. Suppose we have proved it for $x \in A^{(\xi)}$ with $\xi \leq \alpha$ and let $x \in A^{(\alpha+1)}$. Let $x_n \in A^{(\alpha)}$ be such that x_n converges to x. By the inductive hypothesis there are $H^n_m \subseteq F_m$ such that $|H^n_m| \leq 1$ for all $m, n \in \mathbb{N}$ and $x_n \in \overline{\bigcup_m H^n_m}$. Define $D \subseteq \mathbb{N} \times \mathbb{N}$

by

$$(n,m) \in D \Leftrightarrow H^n_m \neq \emptyset$$

By an standard diagonalization procedure it is easy to find $D' \subseteq D$ such that (i) $\{n\} \times \mathbb{N} \,\cap\, D'$ is infinite for all n and (ii) $|\mathbb{N} \times \{m\} \,\cap D'| \leq 1$ for all m. Let

$$F = \bigcup_{(n,m)\in D'} H^n_m$$

Using (i) it follows that $x_n \in \overline{F}$ and thus $x \in \overline{F}$ and from (ii) we get that $|F \cap F_m| \leq 1$. $\square$

Proposition 3.2 *A Frechet point $x \in X$ is a p$^+$-point iff it has the weak diagonal sequence property.*

Proof. Let x be a Frechet point. First, suppose x is a p$^+$-point. Let x_{nk} be a double-indexed sequence in X such that for all n, $x_{nk} \rightarrow_k x$. Let P_n be $\{x_{mk} :\ m \geq n,\, k \geq 1\}$. Then P_n is a decreasing sequence of sets with $x \in \overline{P_n}$. Since x is a p$^+$-point, there is $P \subseteq^* P_n$ such that $x \in \overline{P}$. Since x is Frechet, there is a sequence $\{y_m\}$ in P converging to x. Pick a subsequence y_{m_j} and an increasing sequence of integer $n(j)$ such that $y_{m_j} \in \{x_{n(j),k} :\ k \geq 1\}$

Suppose now that x has the weak diagonal sequence property. To see that x is a p$^+$-point, let P_n be a decreasing sequence of sets with $x \in \overline{P_n}$. Since x is Frechet, there is $x_{nk} \in P_n$ such that $x_{nk} \rightarrow_k x$ for all n. Let $n(j)$ and $k(j)$ two sequences of integers such that $n(j)$ is increasing and $x_{n(j),k(j)} \rightarrow_j x$. Let P be the range of this sequence. Then $P \subseteq^* P_n$ for all n. $\square$

The following result from [18] (see exercise 3 in page 53) is the key fact for analyzing bisequentiality in the realm of analytic spaces.

Theorem 3.3 *A point in an analytic space is selective if and only if it is bisequential.*

4 Diagonal sequence properties

It turns out that the diagonal sequence property is quite strong in the context of analytic Frechet spaces. An interpretation of the analytic gap theorem of [17] shows that a Frechet analytic space has the diagonal sequence property iff it is first countable (see also [20, theorem 6.6]). We now show that a similar interpretation of the analytic gap theorem of [17] gives an analogous result for the class of analytic sequential rather than Frechet spaces.

Theorem 4.1 *An analytic sequential space X is weakly first countable if and only if X has the diagonal sequence property.*

Proof. Suppose X has the diagonal sequence property. Let $C_x = \{A \subseteq X :\ A \to x\}$ and $D_x = \{B \subseteq X :\ x \notin \overline{B}\}$. Note that C_x and D_x are two orthogonal families of subsets of ω, i.e., the intersection of an arbitrary member of C_x with an arbitrary member of D_x is finite. Note also that D_x is an analytic family of subsets of ω so the analytic gap theorem of [17] applies giving us the following two alternatives written in the terminology of [17]:

(1) There is a sequence $A(x,n)$ of members of $C_x^{\perp}$ such that for all $B \in D_x$ there is m such that $B \subseteq A(x,m)$.

(2) There is a C_x-tree all of whose branches belong to D_x.

An application of the diagonal sequence property of X easily eliminates the alternative (2). It follows that the alternative (1) holds. Without lost of generality, we may assume that the $A(x,n)$'s are increasing for each x and $x \notin A(x,n)$ for all n. Let $B(x,m) = X \setminus A(x,m)$. We claim that the $B(x,m)$'s form a weak base. In fact, let V be an open set and $x \in V$. Suppose that $B(x,n) \nsubseteq V$ for all n. Then pick $x_n \in B(x,n) \setminus V$ and let $B = \{x_n :\ n \in \mathbb{N}\}$. Since $B \cap V$ is empty, then $x \notin \overline{B}$. Thus by (1) there is m such that $B \cap B(x,m)$ is finite. But this is a contradiction, since the $B(x,n)$'s are decreasing.

Now suppose that a subset V of X satisfies that for all $x \in V$, there is m with $B(x,m) \subseteq V$. We will show that V is sequentially open, and thus open. Let $x \in V$ and $A \in C_x$. By hypothesis there is m such that $B(x,m) \subseteq V$. By (1), $A \cap A(x,m)$ is finite, then $A \subseteq^* B(x,m) \subseteq V$.

Suppose now that X is weakly first countable and $B(x,m)$ is a weak base. Let $d(x,y)$ be $1/m$ if $y \in B(x,m) \setminus B(x,m+1)$ and $d(x,y) = 2$ otherwise. To see that the diagonal sequence property holds it clearly suffices to show that a sequence x_n converges to x iff $d(x,x_n) \to 0$. To show this claim, suppose $x_n \to x$ but $d(x,x_n) \not\to 0$. Let $k > 0$ be such that $A = \{x_n :\ d(x,x_n) > 1/k\}$ is infinite. Then $A \cup \{x\}$ is closed. Hence for all $y \notin A \cup \{x\}$ there is n_y such that $B(y,n_y) \subseteq X \setminus (A \cup \{x\})$. Since $B(x,k) \subseteq X \setminus A$. It follows that $X \setminus A$ is open. This contradicts that x_n converges to x. The other implication is straightforward. □

Corollary 4.2 *The topology of every analytic sequential space with the diagonal sequence property is $F_{\sigma\delta}$.*

Remark 4.3 Note that the typical analytic sequential spaces S_2 and S_ω are weakly first countable and that in some sense this is the way their topologies are given.

Let us now return to analytic Frechet spaces. We have already noted that in this context the diagonal sequence property reduces to first countability. The following result gives a characterization of the weak diagonal sequence property in this context.

Theorem 4.4 *An analytic Frechet space is bisequential if and only if it has the weak diagonal sequence property.*

Proof. Only the implication from the weak diagonal sequence property towards the bisequentiality is not obvious. Suppose X is Frechet and has the weak diagonal sequence property. From propositions 3.1 and 3.2 we have that every point of X is selective. Then by theorem 3.3 X is bisequential. $\square$

Corollary 4.5 *A Frechet analytic space is bisequential iff it contains no closed copy of $S(\omega)$.*

Remark 4.6 The assumption that X is Frechet is essential here. For example, the result does not extend to the wider class of sequential spaces which can be seen by noting that Arens space S_2 contains no copy of $S(\omega)$.

Corollary 4.7 *A countable space with an F_σ basis is bisequential if and only if it is sequential.*

Proof. Suppose X is sequential. Since neither Arens space nor the sequential fan admits a F_σ basis, then X contains no closed copy of neither of them. Thus X is Frechet and has the weak diagonal sequence property. Hence X is bisequential. $\square$

Remark 4.8 There are spaces with a F_σ basis which are not sequential and spaces with F_σ basis which are sequential but not metrizable. For example, it is well known that the space $CO(2^{\mathbb{N}})$ of all clopen subsets of $2^{\mathbb{N}}$, as a subspace of $\{0,1\}^{2^{\mathbb{N}}}$ with the product topology, it is not sequential but as it is easily checked it has a F_σ basis. On the other hand, the subspace $BCO(2^{\mathbb{N}})$ of $CO(2^{\mathbb{N}})$ consisting only of basic clopen sets (including the empty set, of course) is (bi)sequential (see Example 5.6 of [20]).

5 Products

The sequential fan $S(\omega)$ is a typical Frechet space whose square is not Frechet (consider the following subset of $S(\omega)^2$: $Z = \{((m,n),(0,m)) : \; m,n \in \mathbb{N}\}$ and note that $(\infty,\infty) \in \overline{Z}$ while no sequence $((m_k,n_k),(0,m_k))$ in Z converges to (∞,∞).) We start this section with a result which shows that in the context of countable analytic space, $S(\omega)$ is a test space for this phenomenon.

Theorem 5.1 *An analytic space X is bisequential if and only if its square X^2 is Frechet.*

Proof. Suppose X^2 is Frechet. Since $S(\omega)^2$ is not Frechet, then X cannot contain a closed copy of $S(\omega)$. Hence by theorem 4.5 X is bisequential. $\square$

Corollary 5.2 *The square of an analytic Frechet space X is Frechet if and only if X contains no closed copy of the sequential fan $S(\omega)$.*

Remark 5.3 Another example of two Frechet analytic spaces whose product is not sequential is the following. Let $\mathcal{F}$ be the dual filter of $\mathrm{FIN} \times \emptyset$, that is to say, the filter on $\mathbb{N} \times \mathbb{N}$ given by $A \in \mathcal{F}$ iff there is n such that $A \cap (\{m \in \mathbb{N} : \; m < n\} \times \mathbb{N}) = \emptyset$. Let Y be the space $\mathbb{N} \times \mathbb{N} \cup \{\infty\}$ with the topology where every element of $\mathbb{N} \times \mathbb{N}$ is isolated and $\mathcal{F}$ is the nbhd filter of ∞. Then Y is Frechet (in fact, metrizable) and $S(\omega) \times Y$ is not sequential. To see this, consider the diagonal $D \subseteq (\mathbb{N} \times \mathbb{N})^2$ which is sequentially discrete but $(\infty,\infty) \in \overline{D}$. It follows that if X is an analytic Frechet space such that $X \times \mathbb{Q}$ is sequential, then X is bisequential. To see this, suppose towards a contradiction that $S(\omega)$ embeds in X. Let Y be the metrizable space given above. Then Y is homeomorphic to a closed subspace of $\mathbb{Q}$. Thus $X \times \mathbb{Q}$ contains a closed copy of $S(\omega) \times Y$. But this is a contradiction, since $S(\omega) \times Y$ is not sequential.

Let us now turn to the productiveness of the weak diagonal sequence property. Nogura [10] has shown that there exist two countable Frechet spaces X and Y with the weak diagonal sequence property such that $X \times Y$ is neither Frechet nor it has the weak diagonal sequence property. From the results of this paper obtained so far it follows that Nogura's spaces have to be quite noneffective. However, this still leaves the following question unanswered.

Question 5.4 *Is the weak diagonal sequence property productive in the class of analytic spaces?*

We shall now see that this question is closely related to the following test-space problem which is clearly of independent interest.

Problem 5.5 *Show that every analytic space without the weak diagonal sequence property contains a copy of the sequential fan.*

Theorem 5.6 *Suppose X and Y are two analytic spaces with the weak diagonal sequence property. Then $S(\omega)$ does not embed in the product $X \times Y$.*

Proof. The proof is really just an effective version of the proof of Nogura's conjecture given in [19], so we are assuming the reader has a copy of that proof at hand. We start with the assumption that $S(\omega)$ does embed into $X \times Y$ and work towards a contradiction. From our assumption one easily constructs two topologies τ_X and τ_Y on $\omega^2 \cup \{\infty\}$ with ∞ as the only non isolated point such that τ_X and τ_Y both have the weak diagonal sequence property while the topology of $S(\omega)$ is generated by $\tau_X \cup \tau_Y$ as subbasis. For $n \in \omega$, set

$$C_n = \{n\} \times \omega$$

Note that each C_n is a converging sequence in both topologies τ_X and τ_Y. Let

$$\mathcal{A} = \{A \subseteq \omega^2 : \ A \to_{\tau_X} \infty \text{ and } A \cap C_n \text{ is finite for all } n\},$$

$$\mathcal{B} = \{B \subseteq \omega^2 : \ B \to_{\tau_Y} \infty \text{ and } B \cap C_n \text{ is finite for all } n\}.$$

Thus, $\mathcal{A}$ (respectively, $\mathcal{B}$) is the family of all sequences that converge to ∞ relative to τ_X (respectively, relatively to τ_Y) and which are orthogonal to each member of the sequence $\{C_n\}$ of converging sequences. Let

$$\mathcal{X} = \{(A, B) \in \mathcal{A} \times \mathcal{B} : A \cap B = \emptyset\}.$$

We endow $\mathcal{X}$ with the standard separable metric topology induced from 2^{ω^2}. Consider the following subset of the set $\mathcal{X}^{[2]}$ of all unordered pairs of elements of $\mathcal{X}$:

$$\mathcal{K} = \{\{(A, B), (A', B')\} \in \mathcal{X}^{[2]} : (A \cap B') \cup (A' \cap B) \neq \emptyset\}.$$

Note that $\mathcal{K}$ is an open subset of $\mathcal{X}^{[2]}$. Recall the following two alternatives given by the effective version of the Open Coloring Axiom:

Case 1. There is perfect $\mathcal{Y} \subseteq \mathcal{X}$ such that $\mathcal{Y}^{[2]} \subseteq \mathcal{K}$.

Case 2: There is a decomposition

$$\mathcal{X} = \bigcup_{n=0}^{\infty} \mathcal{X}_n$$

such that $(\mathcal{X}_n)^{[2]} \cap \mathcal{K} = \emptyset$ for all n.

The proof of [19] shows that neither of these two alternatives are possible and this would finish the proof if the effective OCA can indeed be applied. Unfortunately, assuming no additional set-theoretic axioms, the effective OCA applies only to analytic base sets $\mathcal{X}$ which is not so in our case here. It turns out that there exist analytic subfamilies $\mathcal{A}^*$ and $\mathcal{B}^*$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, which are large enough in the sense that they contain all the results of the applications of the weak diagonal sequence property needed for in the argument of [19] in order to eliminate the Case 2 for the corresponding analytic family

$$\mathcal{X}^* = \{(A, B) \in \mathcal{A}^* \times \mathcal{B}^* : A \cap B = \emptyset\}.$$

For a given subset C of ω, set $C^{[2]} = \{(m, n) \in C^2 : m < n\}$. Let

$$\mathcal{R} = \{(C, A) \in \mathcal{P}(\omega) \times \mathcal{A} : A \subseteq^* C^{[2]}\},$$
$$\mathcal{S} = \{(C, B) \in \mathcal{P}(\omega) \times \mathcal{B} : B \subseteq^* C^{[2]}\}.$$

The assumption that τ_X and τ_Y have the weak diagonal sequence property yields that $\mathcal{R}$ and $\mathcal{S}$ are two coanalytic relations whose projections on the first coordinate include the family $[\omega]^\omega$ of all infinite subsets of ω. Using a standard Mathias-forcing argument one can show that these two relations admit local uniformizations by continuous functions. In other words, there is $D^* \in [\omega]^\omega$ and continuous functions

$$f : [D^*]^\omega \to \mathcal{P}(\omega^2) \;\; and \;\; g : [D^*]^\omega \to \mathcal{P}(\omega^2)$$

such that $\mathcal{R}(C, f(C))$ and $\mathcal{S}(C, g(C))$ hold for all $C \in [D^*]^\omega$. Finally, we let $\mathcal{A}^*$ and $\mathcal{B}^*$ be the families obtained from the ranges of f and g, respectively, by closing under finite changes. Note that so obtained families $\mathcal{A}^*$ and $\mathcal{B}^*$ are analytic. The reader is left to check that $\mathcal{A}^*$ and $\mathcal{B}^*$ are indeed large enough to allow the proof from [19] be applied in showing that $\mathcal{X}^*$ cannot be covered by countably many sets whose symmetric squares avoid $\mathcal{K}$. $\square$

6 Homogeneous spaces

Let $\vec{\mathcal{F}} = \{\mathcal{F}_s : \ s \in \omega^{<\omega}\}$ be a collection of filters over $\mathbb{N}$ each of which contains every cofinite sets. Define a topology $\tau_{\vec{\mathcal{F}}}$ over $\omega^{<\omega}$ by letting a subset U of $\omega^{<\omega}$ be open if and only if

$$\{n \in \mathbb{N} : \ s\widehat{\ }n \in U\} \in \mathcal{F}_s \text{ for all } s \in U$$

It is clear that $\tau_{\vec{\mathcal{F}}}$ is T_2, zero dimensional and has no isolated points. If all filters $\mathcal{F}_s$ are equal to a filter $\mathcal{F}$ we will denote by $\tau_{\mathcal{F}}$ the corresponding topology. A particular important case is when $\mathcal{F}$ is the filter CF of all cofinite sets. It is not difficult to show that the corresponding space is homeomorphic to S_ω. To analyze this class of spaces we need the following notion. An *$\vec{\mathcal{F}}$-tree with root $s \in \omega^{<\omega}$* is a subset T of $\omega^{<\omega}$ with s as its minimal node such that

(i) If $t \in T$, then $t|m \in T$ for all $lh(s) \leq m \leq lh(t)$.

(ii) $\{n : \ t\widehat{\ }n \in T\} \in \mathcal{F}_t$, for all $t \in T$.

Proposition 6.1 *The collection of all $\vec{\mathcal{F}}$-trees with root s forms a local basis at s for the topology $\tau_{\vec{\mathcal{F}}}$.*

Proof. It is clear that every $\vec{\mathcal{F}}$-tree is a $\tau_{\vec{\mathcal{F}}}$-open set. So, it suffices to show that for every $\tau_{\vec{\mathcal{F}}}$-open set O and any $s \in O$ there is a $\tau_{\vec{\mathcal{F}}}$-tree T with stem s contained in O. We will define by induction a sequence $T_n \subseteq \omega^{lh(s)+n}$ such that $T_n \subseteq O$ for all n. For $t \in \omega^{<\omega}$, let $A_t = \{n : \ t\widehat{\ }n \in O\}$. Since O is open, $A_t \in \mathcal{F}_t$ for every $t \in O$. Let $T_0 = \{s\}$. Suppose we have defined T_k and let

$$T_{k+1} = \bigcup_{t \in T_k} \{t\widehat{\ }n : \ n \in A_t\}$$

By the inductive hypothesis, $T_{k+1} \subseteq O$ and therefore $A_t \in \mathcal{F}_t$ for every $t \in T_{k+1}$. Let $T = \bigcup_k T_k$. Then T is a $\tau_{\vec{\mathcal{F}}}$-tree with stem s and by construction $T \subseteq O$. $\square$

A particular striking class of examples is obtained when one takes each $\mathcal{F}_s$ to be an ultrafilter. In this case $\tau_{\mathcal{F}}$ is an extremely disconnected topology of $\omega^{<\omega}$ so this case resembles the space of Sirota [13]. Note that so extremal choice of the parameter is not going to give us anything new in the realm of analytic spaces since every analytic extremely disconnected space must be discrete. To see this, suppose X is an extremely disconnected non discrete analytic space. Let $x \in X$ be a non isolated point. Let $\{O_i\}_i \in I$ be a maximal family of pairwise disjoint open sets such that $x \notin \overline{O_i}$ for all $i \in I$. Define an ideal $\mathcal{I}$ over I by letting $A \in \mathcal{I}$ if and only if $x \notin \overline{\bigcup_{i \in A} O_i}$. Then $\mathcal{I}$ is a non principal analytic ideal on I. It is also easy to check that it is maximal. However, it is well-known that nonprincipal maximal ideals are never analytic.

It is easy to see that $\tau_{\mathcal{F}}$ is never a Frechet topology. For instance, observe that $\emptyset$ is an accumulation point of ω^2, but no sequence in ω^2 converges to it. Recall that a filter $\mathcal{F}$ over $\mathbb{N}$ is said to be *Frechet* if for all $A \subseteq \mathbb{N}$ that has nonempty intersection with every member of $\mathcal{F}$, there is an infinite $B \subseteq A$ such that $B \setminus C$ is finite for all $C \in \mathcal{F}$ (or equivalently, the space $\mathbb{N} \cup \{\infty\}$ with the topology where each $n \in \mathbb{N}$ is isolated and $\mathcal{F}$ is the nbhd filter of ∞ is a Frechet space). The next proposition characterizes when $\tau_{\bar{\mathcal{F}}}$ is sequential.

Proposition 6.2 *The topology $\tau_{\bar{\mathcal{F}}}$ is sequential if and only if $\mathcal{F}_s$ is Frechet for every $s \in \omega^{<\omega}$.*

Proof. Suppose that $\tau_{\bar{\mathcal{F}}}$ is sequential. Fix $s \in \omega^{<\omega}$. It is clear that $X_s = \{s\} \cup \{s^\frown n : n \in \mathbb{N}\}$ is a closed subspace of $\omega^{<\omega}$ and thus it is itself sequential. Since s is the only non isolated point of X_s, then it is Frechet. Thus $\mathcal{F}_s$ is Frechet as it is the nbhd filter of s in X_s . Conversely, suppose that each $\mathcal{F}_s$ is Frechet and let $A \subseteq \omega^{<\omega}$. We will show that the sequential closure $[A]_{\text{seq}}$ of A is $\tau_{\bar{\mathcal{F}}}$-closed. Let $t \notin [A]_{\text{seq}}$. We claim that the set $D = \{n \in \mathbb{N} : t^\frown n \notin [A]_{\text{seq}}\}$ belongs to $\mathcal{F}_t$. Otherwise, if $D \notin \mathcal{F}_t$, then $E = \mathbb{N} \setminus D$ satisfies that $E \cap C \neq \emptyset$ for every $C \in \mathcal{F}_t$. Therefore, as $\mathcal{F}_t$ is Frechet, there is $B \subseteq E$ such that $B \setminus C$ is finite for all $C \in \mathcal{F}_t$. This says that $\{t^\frown n : n \in B\}$ is a sequence in $[A]_{\text{seq}}$ converging to t, which contradicts our assumption. Thus D belongs to $\mathcal{F}_t$ and therefore $[A]_{\text{seq}}$ is closed. $\square$

Corollary 6.3 *If $(\omega^{<\omega}, \tau_{\mathcal{F}})$ is sequential, then S_ω embeds into it as a closed subspace.*

Question 6.4 *Is there a homogeneous analytic sequential space of sequential order ω_1 containing no copy of S_ω ?*

The following simple fact shows that many of the spaces from the class are indeed homogeneous.

Proposition 6.5 *For every filter $\mathcal{F}$, the space $(\omega^{<\omega}, \tau_{\mathcal{F}})$ is homogeneous.*

Proof. Since $(\omega^{<\omega}, \tau_{\mathcal{F}})$ is a regular space without isolated points, it suffices to show (see [4]) that for every $s, t \in \omega^{<\omega}$ there are two $\tau_{\mathcal{F}}$-clopen nbhds U and V, respectively, of s and t, and a homeomorphism $h : U \to V$ with $h(s) = t$. Let U and V be the $\mathcal{F}$-trees with stem s, respectively t, $\{u \in \omega^{<\omega} : s \leq u\}$ and $\{u \in \omega^{<\omega} : t \leq u\}$. Define h by $h(s\frown u) = t\frown u$. It is easy to check that h is an homeomorphism. □

For two filters $\mathcal{F}$ and $\mathcal{G}$ on ω we write $\mathcal{F} \leq \mathcal{G}$ if there is A in $\mathcal{F}$ and B in $\mathcal{G}^+$ such that the restriction of $\mathcal{F}$ on A and the restriction of $\mathcal{G}$ on B are isomorphic filters. Note that if $(\omega^{<\omega}, \tau_{\mathcal{F}})$ is homeomorphic to a subspace of $(\omega^{<\omega}, \tau_{\mathcal{G}})$ then $\mathcal{F} \leq \mathcal{G}$.

Proposition 6.6 *There is a family $\mathcal{F}_i$ $(i \in I)$ of size bigger than the continuum of Frechet filters on ω such that $\mathcal{F}_i \not\leq \mathcal{F}_j$ whenever $i \neq j$.*

Proof. We construct the filters on $2^{<\omega}$ rather than on ω identifying the Cantor set 2^ω with the set of branches of the binary tree $2^{<\omega}$. For a subset C of the complete binary tree $2^{<\omega}$, let $[C] = \{x \in 2^\omega : x \cap C \text{ is infinite}\}$. For a subset X of the Cantor set 2^ω, let $\mathcal{F}_X$ be the filter on $2^{<\omega}$ generated by the complements of branches from X as well as complements of finite sets. Note that $\mathcal{F}_X$ is always a Frechet filter on $2^{<\omega}$. Suppose $\phi : A \to B$ is a bijection witnessing $\mathcal{F}_X \leq \mathcal{F}_Y$ for some subsets X and Y of the Cantor set. Define $f_\phi : [A] \to \mathcal{P}([\mathcal{B}])$ and $g_\phi : [B] \to \mathcal{P}([\mathcal{A}])$ as follows

$$f_\phi(x) = \{y \in 2^\omega : \phi''x \cap y \text{ is infinite}\},$$
$$g_\phi(y) = \{x \in 2^\omega : \phi^{-1}y \cap x \text{ is infinite}\}.$$

Note that $f_\phi(x)$ is a finite subset of Y for all $x \in X$ and that $g_\phi(y)$ is a finite subset of X for all $y \in Y$. Also note that $y \in f_\phi(x)$ iff $x \in g_\phi(y)$. Fix enumerations $\{x_\eta\}_{\eta<c}$ of $2^{\mathbb{N}}$ and $\{\phi_\alpha\}_{\alpha<c}$ of all bijections between two subsets of $2^{<\omega}$. Using a standard diagonalisation argument construct a subset $Z = \{z_\alpha : \alpha < c\}$ such that for every $\beta, \eta < \alpha < c$, if $f_{\phi_\beta}(z_\eta)$ is finite, then z_α is above (w.r.t. the enumeration of $2^{\mathbb{N}}$) every element of $f_{\phi_\beta}(z_\eta)$; and analogously for g_{ϕ_β}.

Pick a family $\mathcal{X}$ of subsets of Z such that $X \setminus Y$ has size continuum for every pair X and Y of distinct elements of $\mathcal{X}$. Then $\mathcal{F}_X$ $(X \in \mathcal{X})$ satisfies the conclusion of the Proposition. In fact, towards a contradiction, suppose ϕ_β witnesses that $\mathcal{F}_X \leq \mathcal{F}_Y$. Pick $\alpha > \beta$ such that $z_\alpha \in X \setminus Y$. Noti ce that $f_{\phi_\beta}(z_\alpha)$ is a finite subset of Y. Let $z_\eta \in f_{\phi_\beta}(z_\alpha)$. Since $z_\alpha \notin Y$, then $\eta < \alpha$ by construction. But $z_\alpha \in g_{\phi_\beta}(z_\eta)$ and this contradict the choice of z_α.

□

Corollary 6.7 *There is a family X_i $(i \in I)$ of size bigger than the continuum of sequential homogeneous spaces of sequential order ω_1 such that the space X_i is not homeomorphic to a subspace of X_j whenever $i \neq j$.*

Note that Corollary 6.7 gives a very generous positive answer to a question from [3] (see page 319). The question has been actually answered long ago by Kannan[7] though he was able to construct only two new sequential homogeneous spaces. It is interesting, however, that Kannan's spaces are both analytic so one may ask for an analytic analogue of Corollary 6.7. The following fact shows that indeed there is such an analogue.

Proposition 6.8 *There is an uncountable family of pairwise nonhomeomorphic analytic sequential spaces of sequential order ω_1.*

Proof. Going back to the proof of Proposition 6.6. By identifying Z with $2^{\mathbb{N}}$, it suffices to take an uncountable collection $\mathcal{X}$ of closed subsets of $2^{\mathbb{N}}$ such that $X \setminus Y$ is uncountable for every pair X and Y of distinct elements of $\mathcal{X}$ (for instance, take $\mathcal{X}$ to be the collection of sets $\mathcal{P}(A)$ for A belonging to an uncountable almost disjoint family of subset of $\mathbb{N}$). Notice that $\mathcal{F}_X$ is Borel (in fact F_σ) when X is closed and in this case the corresponding topology $\tau_{\mathcal{F}_X}$ is analytic (in fact $F_{\sigma\delta}$). $\square$

Applying the result of van Douwen [4] to the homogeneous space $(\omega^{<\omega}, \tau_{\mathcal{F}})$, we conclude that for every $\mathcal{F}$ and every countable and infinite group G there is a topology τ on G such that (G, τ) is homeomorphic to $(\omega^{<\omega}, \tau_{\mathcal{F}})$ and such that the multiplication of G is left-continuous with respect to the topology τ. So, it is natural to ask whether one can find such a topology τ for which the group operations are actually continuous. It turns out that there is no group structure on $\omega^{<\omega}$ compatible with $\tau_{\mathcal{F}}$ for $\mathcal{F}$ Frechet. In fact, one can say even more.

Theorem 6.9 *If a topological group is homeomorphic to a space of the form $(\omega^{<\omega}, \tau_{\mathcal{F}})$, then it has no non trivial convergent sequences.*

Proof. Let $*$ and $^{-1}$ be some group operations on $\omega^{<\omega}$ continuous relative to some topology on $\omega^{<\omega}$ of the form $\tau_{\mathcal{F}}$ for a filter $\mathcal{F}$ on $\mathbb{N}$. We can assume w.l.o.g that $\emptyset$ is the identity element. Suppose $x_k \to x$ is a non trivial convergent sequence in $(\omega^{<\omega}, \tau_{\mathcal{F}})$. Also w.l.o.g. assume that $x = \emptyset$. Observe that any sequence $\tau_{\mathcal{F}}$-convergent to $\emptyset$ is eventually of the form $\langle n_k \rangle$ for some strictly increasing sequence $n_k \in \mathbb{N}$. Thus we can assume that the x_k's are of that form. Now for each fixed k, we have that $\langle n_j \rangle * \langle n_k \rangle \to \langle n_k \rangle$. The only non trivial sequences converging to $\langle n \rangle$ relative to $\tau_{\mathcal{F}}$ are eventually of the form $\langle n, m_k \rangle$ for some strictly increasing sequence $m_k \in \mathbb{N}$. Therefore we can find a strictly increasing sequence j_k of integers such that $y_k = \langle n_{j_k} \rangle * \langle n_k \rangle$ belongs to ω^2 for all k. By the joint continuity of the operations y_k converges to $\emptyset$. This is a contradiction, since no sequence in ω^2 converges to $\emptyset$. $\square$

Remark 6.10 It is well known, that if $\mathcal{U}$ is a selective ultrafilter, then $([\omega]^{<\omega}, \tau_{\mathcal{U}})$ is a topological group with the operation of symmetric difference (see [13]). By theorem 6.9, $([\omega]^{<\omega}, \tau_{\mathcal{U}})$ contains no nontrivial convergent sequences.

7 Topological groups

It is known that there exist analytic homogeneous sequential spaces with an arbitrary sequential order $\leq \omega_1$ (see [5]). It turns out that for topological groups the situation is much less clear. While there exists a countable (analytic) sequential group of sequential order ω_1 (the free group of the converging sequence), the existence of such group of sequential order $< \omega_1$ depends presently on the Continuum Hypothesis(see, [12]). It is therefore rather natural to investigate the following Question.

Question 7.1 *Is every analytic sequential group of countable sequential order metrizable?*

This can also be considered as a variation on the following well known open problem about countable topological groups (see, e.g., [12]).

Question 7.2 *(Malyhin) Is every countable Frechet topological group metrizable?*

It is well known that the Haar group $\{0,1\}^{\omega_1}$ may be a Frechet space under various additional set-theoretic assumptions, so in such a situation any countable dense subgroup of $\{0,1\}^{\omega_1}$ gives a negative answer to Malyhin's problem (see [12]). Perhaps less known is the fact that only Martin's axiom is sufficient to produce not only negative answer to Malyhin's problem but also to the problem of productiveness of the Frechet property in the realm of countable topological groups. This can easily be deduced from the fact (first pointed out in the Remark on p.150 of [16]) that under MA there exist two sets of reals X and Y such that $C_p(X)$ and $C_p(Y)$ are Frechet spaces but their product is not. To see this, pick a countable subset D of the product which accumulates to 0 but no sequence of elements of D

converges to 0, and let G and H be the subgroups of $C_p(X)$ and $C_p(Y)$, respectively, generated by the projections of D. It is easily seen that the topologies of such subgroups G and H can never be analytic. Similarly, no countable dense subgroup of $\{0,1\}^{\omega_1}$ can have analytic topology. So, it is natural to consider the status of Malyhin's problem in the class of analytic spaces. As an application of theorem 4.4, we give a positive answer to the effective version of Malyhin's problem.

Theorem 7.3 *A countable Frechet topological group is metrizable iff its topology is analytic.*

Proof. Only the reverse implication needs a proof. Let G be a Frechet topological group with an analytic topology. It suffices to show that G is first countable. By theorems 2.4 and 4.4, G is bisequential. It is known that bisequential groups are first countable (see [12]). To see this, let $\mathcal{U}$ be an ultrafilter extending the nbhd filter of the identity element e of G and moreover assume that $\mathcal{U}$ contains no nowhere dense sets. Let A_n be a sequence of elements of $\mathcal{U}$ converging to e. Let $B_n = int(\overline{A_n})$. Note that B_n is nonempty for all n. Moreover, note that $B_n \cdot (B_n)^{-1}$ form a countable nbhd base of e. □

Remark 7.4 There are examples of sequential analytic topological groups without the Frechet property. For example, the free topological group over the convergent sequence is sequential but not Frechet and its topology is analytic (see [12]).

References

[1] R. Arens, Note on convergence in topology, *Math. Mag.*, 23:229–234, 1950.

[2] S. Argyros, G. Godefroy and H. P. Rosenthal, Descriprtive Set Theory and Banach Spaces, in *Handbook of Banach Spaces,* vol 2, North Holland, Amsterdam, to appear.

[3] A. V. Arkhanglel'skiĩ and S. P. Franklin, Ordinal invariants for topological spaces, *Mich. Math. J.*, 15:313–320, 1968.

[4] E. K. van Douwen, Countable homogenous spaces and countable groups, in Z. Frolik et al., editor, *Proc. Sixth Prague Topological Symposium 1986*, pages 135–154, Helderman Verlag, 1988.

[5] L. Foged, Sequential order of homogeneous spaces and product spaces, *Topology Proc.*, 6:287-298, 1981.

[6] D. H. Fremlin, Sequential convergence in $C_p(X)$, *Comment. Math. Univ. Carolin.*, 35:371-382, 1994.

[7] V Kannan, An extension that nowhere has the Frechet property, *Michigan Math. J.*, 20:225-234, 1973.

[8] A. R. D. Mathias, Happy families, *Ann. Math. Logic*, 12:59-111, 1977.

[9] E. Michael, A quituple quotient quest, *Top. and its Appl.*, 2:91–138, 1972.

[10] T. Nogura, *Products of $\langle \alpha_i \rangle -$spaces, Topology and its Appl.*, 21, 251-259, 1985.

[11] P. Nyikos, Metrizability and the Frechet-Urysohn property in topological groups, *Proc. Amer. Math. Soc.*, 83(4):793–801, 1981.

[12] D. Shakhmatov, Convergence in the presence of algebraic structure, in M. Husek and J. van Mill (editors), *Recent Progress in General Topology II*, to appear.

[13] S. Sirota, A product of topological groups and extremal disconnectedness, *Mat. Sb.(N,S.)*, 97(121):179-192, 1969.

[14] F. Siwiec, Generalizations of first axiom of countability, *Rocky Mountain J. Math.*, 5:1–60, 1975.

[15] N. Steenrod, A convenient category of spaces, *Mich. Math, J.*, 15:133-152, 1967.

[16] S. Todorcević, Some applications of S and L combinatorics, *Ann. New York Acad. Sci.*, 705, 130-167, 1993.

[17] S. Todorčević, Analytic gaps, *Fund. Math.*, 150(1):55–66, 1996.

[18] S. Todorčević, *Topics in Topology*, LNM 1652. Springer, 1997.

[19] S. Todorcević, *A proof of Nogura's conjecture*, preprint 2002.

[20] S. Todorčević and C. Uzcátegui, Analytic topologies over countable sets, *Top. and its Appl.*, 111(3):299–326, 2001.

Université Paris VII-C.N.R.S., UMR 7056, 2 Place Jussieu, 75251 Paris Cedex 05, France.
stevo@math.jussieu.fr

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida 5101, Venezuela.
uzca@ciens.ula.ve