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A Self-similar Dynamics in Viscous Spheres

W. Barreto,[1,2] J. Ovalle[3,4] and B. Rodríguez[1]

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We study the evolution of radiating and viscous fluid spheres assuming an additional homothetic symmetry on the spherically symmetric spacetime. We match a very simple solution to the symmetry equations with the exterior one (Vaidya). We then obtain a system of two ordinary differential equations which rule the dynamics, and find a self-similar collapse which is shear-free and with a barotropic equation of state. Considering a huge set of initial self-similar dynamics states, we work out a model with an acceptable physical behavior.

KEY WORDS : Homothetic symmetry ; barotropic equation of state

1. INTRODUCTION

Often many authors assume spherical symmetry and perfect fluid approximation to deal with the problem of self-gravitating and collapsing distributions of matter. Also, they use extensively progressive waves or similarity solutions (see Refs. 1,2 and references therein). If the fluid is perfect the only equation of state compatible with self-similar fluids is the barotropic one [2]. The present paper deals in part with the validity of the barotropic equation of state for a viscous and radiating fluid sphere.

[1] Laboratorio de Física Teórica, Departamento de Física, Escuela de Ciencias, Núcleo de Sucre, Universidad de Oriente, Cumaná, Venezuela.
E-mail: wbarreto@cumana.sucre.udo.edu.ve
[2] Centro de Astrofisica Teórica, Universidad de los Andes, Mérida, Venezuela
[3] Departamento de Física, Escuela de Física y Matematicas, Facultad de Ciencias,Universidad Central de Venezuela, Caracas, Venezuela
[4] Departamento de Física, Universidad Simón Bolívar, Caracas, Venezuela

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In general, there are two types of self-similar space-times depending on whether they are invariant or not under scale transformations. Scale-free self-similar solutions are the similarity solutions of type one and the resulting space-time admits homothetic Killing vectors. Type two similarity solutions are not invariant under the simple scaling group [3–6]. The self-similar symmetry has been reported to characterize these two types of self-similar space-times [7].

Spherically symmetric and homothetic space-times show naked singularities. The assumption of similarity rather than spherical symmetry is crucial in determining the nature of the singularity in any gravitationally collapsing configuration [8,9]. So far, self-similar space-times have been studied mainly in cosmological contexts [10–15].

Considering that the perfect fluid approximation is likely to fail, at least in some stages of stellar collapse, in this paper we study radiating and viscous fluid spheres. Specifically, we have been concerned with the radiative shear viscosity and its effect on the gravitational collapse [16–18]. We do not consider here the temperature profiles to determine which processes can take place during the collapse. For this purpose, transport equations have been proposed to avoid pathological behaviors (see for instance Ref. 19 and references therein). The motivation of this work was a recent study of radiating and dissipative spheres [20]. We assume an additional symmetry (homothetic motion) within the viscous fluid sphere without heat flow in the streaming out limit.

The organization of this paper is the following. Section 2 shows the field equations, the junction conditions and the surface equations. In Section 3 we write the homothetic motion equations in a convenient form. We propose a very simple solution in Section 4 to work out some models. Finally, in Section 5, we draw conclusions.

2. DYNAMICS AND MATCHING

2.1. Field equations

To write the Einstein field equations we use the line element in Schwarzschild-like coordinates

$$ds^2 = e^{\upsilon}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

where $\upsilon = \upsilon(t, r)$ and $\lambda = \lambda(t, r)$, with $(t, r, \theta, \phi) \equiv (0, 1, 2, 3)$.

In order to get the physical input we introduce the Minkowski coordinates (τ, x, y, z) by [21]

$$d\tau = e^{\upsilon/2}dt, \quad dx = e^{\lambda/2}dr, \quad dy = r\,d\theta, \quad dz = r\sin\theta d\phi, \tag{2}$$

In these expressions υ and λ are constants, because they have only local values.

Next we assume that, for an observer moving relative to these coordinates with velocity ω in the radial (x) direction, the space contains

- a viscous fluid of density ρ, pressure $\hat{p}$, effective bulk pressure p_ζ and effective shear pressure p_η, and
- unpolarized radiation of energy density $\hat{\varepsilon}$.

For this moving observer, the covariant energy tensor in Minkowski coordinates is thus

$$\begin{pmatrix} \rho + \hat{\varepsilon} & -\hat{\varepsilon} & 0 & 0 \\ -\hat{\varepsilon} & \hat{p} + \hat{\varepsilon} - p_\zeta - 2p_\eta & 0 & 0 \\ 0 & 0 & \hat{p} - p_\zeta + p_\eta & 0 \\ 0 & 0 & 0 & \hat{p} - p_\zeta + p_\eta \end{pmatrix}. \tag{3}$$

Note that from (2) the velocity of matter in the Schwarzschild coordinates is

$$\frac{dr}{dt} = \omega e^{(\upsilon - \lambda)/2}. \tag{4}$$

Now, by means of a Lorentz boost and defining $\tilde{p} \equiv \hat{p} - p_\zeta$, $p_r \equiv \tilde{p} - 2p_\eta$, $p_t \equiv \tilde{p} + p_\eta$ and $\varepsilon \equiv \hat{\varepsilon}(1 + \omega)/(1 - \omega)$ we write the field equations in relativistic units ($G = c = 1$) as follows:

$$\frac{\rho + p_r\omega^2}{1 - \omega^2} + \varepsilon = \frac{1}{8\pi r}\left[\frac{1}{r} - e^{-\lambda}\left(\frac{1}{r} - \lambda_{,r}\right)\right], \tag{5}$$

$$\frac{p_r + \rho\omega^2}{1 - \omega^2} + \varepsilon = \frac{1}{8\pi r}\left[e^{-\lambda}\left(\frac{1}{r} + \upsilon_{,r}\right) - \frac{1}{r}\right], \tag{6}$$

$$p_t = \frac{1}{32\pi}\left\{e^{-\lambda}\left[2\upsilon_{,rr} + \upsilon_{,r}^2 - \lambda_{,r}\upsilon_{,r} + \frac{2}{r}(\upsilon_{,r} - \lambda_{,r})\right] - e^{-\upsilon}[2\lambda_{,tt} + \lambda_{,t}(\lambda_{,t} - \upsilon_{,t})]\right\} \tag{7}$$

$$(\rho + p_r)\frac{\omega}{1 - \omega^2} + \varepsilon = -\frac{\lambda_{,t}}{8\pi r}e^{-(1/2)(\upsilon + \lambda)}, \tag{8}$$

where the comma (,) represents partial differentiation with respect to the indicated coordinate. Equations (5)–(8) are formally the same as for an anisotropic fluid in the streaming out approximation.

At this point, for the sake of completeness, we write the effective viscous pressures in terms of the bulk viscosity ζ, the volume expansion

Θ, the shear viscosity η and the scalar shear σ [18]

$$p_\zeta = \zeta\Theta \tag{9}$$

$$p_\eta = \frac{2}{\sqrt{3}}\eta\sigma \tag{10}$$

where

$$\Theta = \frac{1}{(1-\omega^2)^{1/2}}\left[e^{-\upsilon/2}\left(\frac{\lambda_{,t}}{2} + \frac{\omega\omega_{,t}}{1-\omega^2}\right) + e^{-\lambda/2}\left(\frac{\upsilon_{,r}}{2}\omega + \frac{\omega_{,r}}{1-\omega^2} + \frac{2\omega}{r}\right)\right] \tag{11}$$

and

$$\sigma = \sqrt{3}\left(\frac{\Theta}{3} - \frac{e^{-\lambda/2}}{r}\frac{\omega}{\sqrt{1-\omega^2}}\right). \tag{12}$$

We have four field equations for six physical variables (ρ, p, ε, ω, ζ and η) and two geometrical variables (υ and λ). Obviously, we require additional assumptions to handle the problem consistently. First, however, we discuss the matching with the exterior solution and the surface equations that govern the dynamics.

2.2. Junction conditions

We describe the exterior space-time by the Vaidya metric

$$ds^2 = \left(1 - \frac{2\mathcal{M}(u)}{R}\right)du^2 + 2dudR - R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{13}$$

where u is a time-like coordinate so that u = constant represents, asymptotically, null cones open to the future and R is a null coordinate ($g_{RR} = 0$). The relationship between the coordinates (t, r, θ, ϕ) and (u, R, θ, ϕ) is

$$u = t - r - 2\mathcal{M}\ln\left(\frac{r}{2\mathcal{M}} - 1\right), \qquad R = r. \tag{14}$$

The exterior and interior solutions are separated by the surface $r = a(t)$. To match both regions on this surface we require the Darmois junction conditions. Thus, demanding the continuity of the first fundamental form, we obtain

$$e^{-\lambda_a} = 1 - \frac{2\mathcal{M}}{R_a} \tag{15}$$

and

$$\upsilon_a = -\lambda_a . \tag{16}$$

From now on the subscript a indicates that the quantity is evaluated at the surface. Now, instead of writing the junction conditions as usual, we demand the continuity of the first fundamental form and the continuity of the independent components of the energy-momentum flow. This last condition guarantees absence of singular behaviors on the surface. It is easy to check that [18,22]

$$\hat{p}_a = p_{\zeta_a} + 2p_{\eta_a}, \tag{17}$$

which expresses the discontinuity of the radial pressure in presence of viscous processes.

2.3. Surface equations

To write the surface equations we introduce the mass function m by means of

$$e^{-\lambda(r,t)} = 1 - 2m(r,t)/r. \tag{18}$$

Substituting (18) into (5) and (8) we obtain, after some arrangements,

$$\frac{dm}{dt} = -4\pi r^2 \left[\frac{dr}{dt} p_r + \varepsilon(1-\omega)(1-2m/r)^{1/2} e^{\upsilon/2} \right]. \tag{19}$$

This equation shows the energetics across the moving boundary of the fluid sphere. Evaluating (19) at the surface and using the boundary condition (17) (which is equivalent to $p_{r_a} = 0$), the energy loss is given by

$$\dot{m}_a = -4\pi a^2 \varepsilon_a (1 - 2m_a/a)(1 - \omega_a). \tag{20}$$

Hereafter overdot indicates d/dt. The evolution of the boundary is governed by eq. (4) evaluated at the surface

$$\dot{a} = (1 - 2m_a/a)\omega_a . \tag{21}$$

Scaling the total mass m_a, the radius a and the time-like coordinate by the initial mass $m_a(t=0) \equiv m_a(0)$,

$$A \equiv a/m_a(0), \quad M \equiv m_a/m_a(0), \quad t/m_a(0) \rightarrow t,$$

and defining

$$F \equiv 1 - \frac{2M}{A}, \tag{22}$$

$$\Omega \equiv \omega_a , \tag{23}$$

$$E \equiv 4\pi a^2 \varepsilon_a (1 - \Omega), \tag{24}$$

the surface equations can be written as

$$\dot{A} = F\Omega, \tag{25}$$

$$\dot{F} = \frac{F}{A}[(1 - F)\Omega + 2E]. \tag{26}$$

Equations (25) and (26) are general within spherical symmetry. We need a third surface equation to specify the dynamics completely for any set of initial conditions and a given luminosity profile $E(t)$. For this purpose we can use eq. (7) or appeal to the conservation equation $T^{\mu}_{1;\mu} = 0$ evaluated at the surface. But we follow here another route, that is, we assume that the space-time admits a one-parameter group of homothetic motion generated by a homothetic Killing vector orthogonal to the four-velocity. These assumptions introduce some restrictions on the surface equations as is shown in the next section.

3. HOMOTHETIC MOTION

We assume that the spherically symmetric space-time within the fluid admits a one-parameter group of homothetic motions. In general, a global vector field ξ on the manifold is called homothetic if $\pounds_{\xi}\mathbf{g} = 2n\mathbf{g}$ holds on a local chart, where n is a constant on the manifold, and $\pounds$ denotes the Lie derivative operator. If $n \neq 0$, ξ is called proper homothetic and it can always be scaled so to have $n = 1$; if $n = 0$ the ξ is a Killing vector on the manifold [23–25]. So, after a constant rescaling we write

$$\pounds_{\xi}\mathbf{g} = 2\mathbf{g}, \tag{27}$$

where the vector field ξ has the general form

$$\xi = \Lambda(r, t)\partial_t + \Gamma(r, t)\partial_r . \tag{28}$$

After simple manipulations we obtain from (27)

$$\Gamma = r, \tag{29}$$

$$\Lambda_{,r} = 0, \tag{30}$$

$$\Lambda m_{,t} + \Gamma m_{,r} = m, \tag{31}$$

$$\Lambda\upsilon_{,t} + \Gamma\upsilon_{,r} + 2\dot{\Lambda} = 2. \tag{32}$$

We further assume that the four-velocity is orthogonal to the orbit of the group

$$\omega = \frac{\Lambda}{r}e^{(\upsilon - \lambda)/2}. \tag{33}$$

Thus we obtain a connection between the time-like component of the homothetic Killing vector and the surface variables,

$$\Lambda(t) = \frac{a\Omega}{F}. \tag{34}$$

Now, expanding υ near the surface, using (15), (16), (34), and evaluating at $r = a$ the equations (5), (8), (31) and (32), after straightforward manipulations we find the surface equation

$$\dot{\Omega} = \frac{(1 - \Omega^2)}{2A}(3F - 1 - 2E). \tag{35}$$

From now on we disregard the bulk effective pressure to promote algebraic consistence.

4. MODELING

In order to work out models we define the self-similar variables

$$X = \frac{m}{r} \tag{36}$$

and

$$Y = \frac{\Lambda}{r} e^{\upsilon/2}. \tag{37}$$

Thus, eqs. (31) and (32) read

$$\Lambda X_{,t} + r X_{,r} = 0 \tag{38}$$

and

$$\Lambda Y_{,t} + r Y_{,r} = 0. \tag{39}$$

In general these equations have solutions $X = X(\varsigma)$ and $Y = Y(\varsigma)$, where ς is

$$\varsigma = re^{-\int dt/\Lambda}. \tag{40}$$

We propose the specific solutions

$$X = C_1 \varsigma^k \tag{41}$$

and

$$Y = C_2 \varsigma^l, \tag{42}$$

where C_1, C_2, k and l are constants.

Solutions (41) and (42) are restricted by (15) and (16). Therefore the geometrical variables are

$$m = m_a \left(\frac{r}{a} \right)^{k+1}, \tag{43}$$

$$e^{\upsilon} = F \left(\frac{r}{a} \right)^{2(l+1)}. \tag{44}$$

In order to get the unique luminosity

$$E = \tfrac{1}{2}[F(k + 2l + 3) - (k + 1)] \tag{45}$$

we use eqs. (5), (6), (43) and (44) together with the boundary conditions (15), (16) and (17) to find

$$\Omega = \frac{2E \pm Z}{2(k + 1)(F - 1)}, \tag{46}$$

where

$$Z = [F^2(5k^2 + 4kl + 10k + 4l^2 + 12l + 9) - 2F(k + 1)(5k + 2l + 3) + 5k^2 + 6k + 1]^{1/2}. \tag{47}$$

Note that "+" in the numerator of (46) represents the collapsing solution and "−" an expanding one. We consider here only Ω^+ situations.

Now, combining eqs. (35) and (46) we obtain an equation $f(F, k, l) = 0$, which is too lengthy to present here, but which permits us to model different situations. The first one is the shear-free and self-similar collapse for which $k = l = 0$, $m/a \approx 0.3096$ (m and a are linear with time) and $\tilde{p} = 0$ at any space-time point. The second possibility appears upon solving for $l = l(F(t = 0), k)$ and includes the previous case. For $k \neq 0$ we obtain shearing models but the homothetic symmetry is broken for $t > 0$.

We work out a "tricky" third scenario by "forgetting" the origin of parameter l, proposing that it depends on time in a very special way. If we imagine N initial self-similar states which represents the history of the collapsing surface, the symmetry equations (31) and (32) are satisfied at every point of the space-time without taking into account the variation with time of l. Therefore, we integrate numerically only eqs. (25) and (26), with (45), (46) and with $l = l(t)$. Here we use standard Runge-Kutta (fourth order) methods and the initial conditions

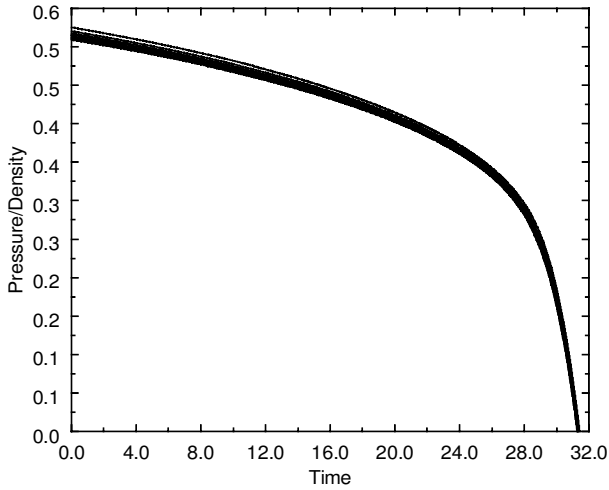

Figure 1. $\tilde{p}/\rho$ as a function of time, for different values of r/a: 0.1 (uppermost curve), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 (lowermost curve).

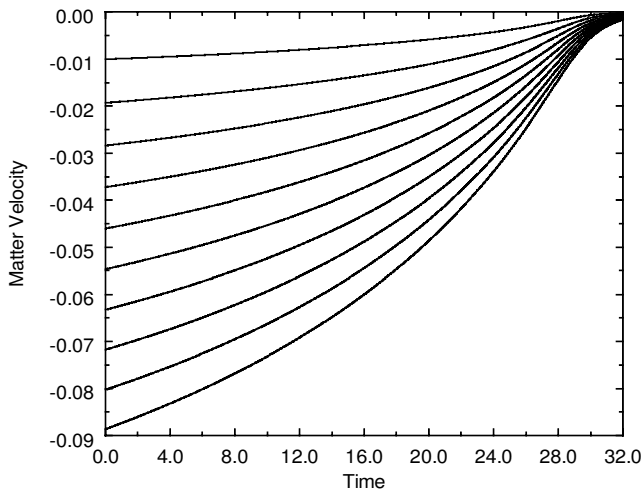

Figure 2. dr/dt as a function of time, for different values of r/a: 0.1 (uppermost curve), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 (lowermost curve).

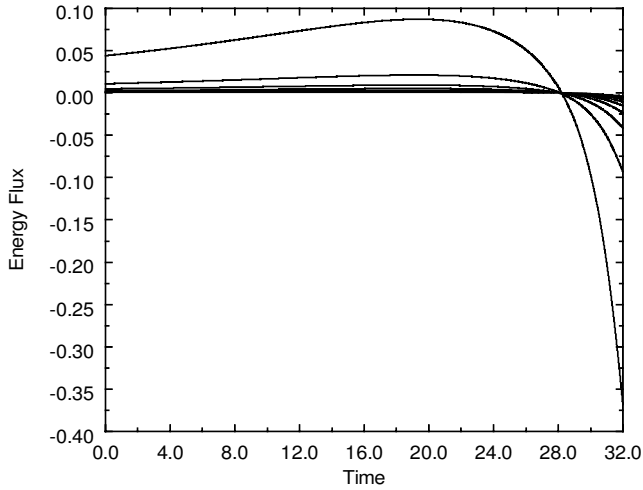

Figure 3. ε as a function of time, for different values of r/a: 0.1 (initially uppermost curve), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 (initially lowermost curve).

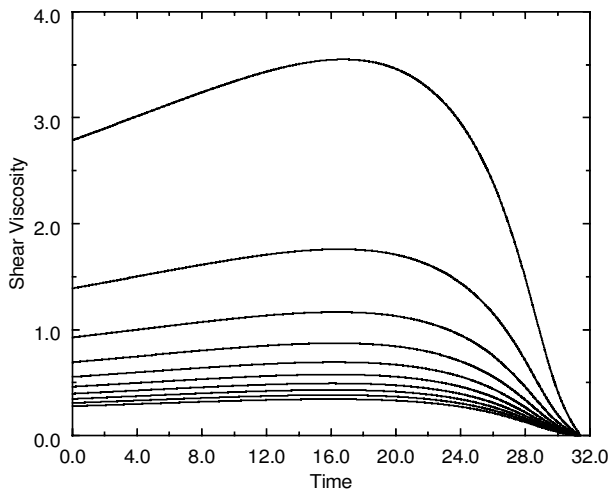

Figure 4. η as a function of time, for different values of r/a: 0.1 (uppermost curve), 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 (lowermost curve).

$$A(0) = 3.255, \qquad F(0) \approx 0.3856.$$

Once the boundary evolution and its energetics are determined, we use (43) [or (18)] and (44) to calculate the physical variables from the field equations. Figures 1–4 sketch the ratio $\tilde{p}/\rho$, dr/dt, ε and η, respectively, for $k = (2)10^{-3}$. These self-similar spheres do not have a barotropic equation of state (Figure 1). All shells evolve with decreasing collapsing velocities (Figure 2). This behavior seems to be connected with the absorption of energy shown in Figure 3 in the late stage. Shear viscosity increases initially with collapse but later decreases with time on any shell.

5. CONCLUSIONS

We have assumed an additional symmetry to the space-time, homothetic motion, to generate non-static and simple solutions. These solutions were matched with the Vaidya one. We found that self-similar spheres with a barotropic equation of state ($\tilde{p} = 0$) are shear-free, this result is in complete accord with theoretical expectation [2,26,27]. Other self-similar scenarios are possible as well if we assume the evolution of the surface as a huge set of initial self-similar states. The shear viscosity profiles obtained in this work coincide qualitatively surprisingly well with others calculated in a more realistic framework [19].

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