Why does gravitational radiation produce vorticity?

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Abstract
We calculate the vorticity of worldlines of observers at rest in a Bondi–Sachs frame, produced by gravitational radiation, in a general Sachs metric. We claim that such an effect is related to the super-Poynting vector, in a similar way as the existence of the electromagnetic Poynting vector is related to the vorticity in stationary electrovacuum spacetimes.

PACS numbers: 04.20.−q, 04.20.Cv, 04.30.Nk

1. Introduction

The theoretical description and the experimental observation of gravitational radiation are among the most relevant challenges confronting general relativity.

A great deal of work has been done so far in order to provide a consistent framework for the study of such a phenomenon. Also, important collaboration efforts have been carried on, and are now under consideration, to put in evidence gravitational waves.

Therefore, it is clear that any specific phenomenon caused by gravitational radiation, and which might be observed, could in principle lead to the detection of such elusive waves and is of utmost relevance.

In this respect, particular attention should be given to the link between gravitational radiation and vorticity of worldlines of observers at rest in a Bondi spacetime [1–3]. Specifically, it has been shown that the leading term in the vorticity (in an expansion of powers of $1/r$) is expressed through the news function in such a way that it will vanish if and only if there is no news (no radiation). This suggests the possibility of detecting gravitational waves by means of gyroscopes [1, 4].

The issue we want to address here is: what is the mechanism by means of which gravitational radiation produces vorticity?

To answer to such a question, it is worth recalling a result obtained by Bonnor [5] concerning the dragging of inertial frames by a charged magnetic dipole. To explain the appearance of vorticity in such spacetimes, Bonnor notes that the corresponding electromagnetic Poynting vector has a non-vanishing component, describing a flow of

electromagnetic energy round in circles where frame dragging occurs [6]. He then suggests that such a flow of energy affects inertial frames by producing vorticity of congruences of particles, relative to the compass of inertia. This conjecture has been recently confirmed to be valid for general axially-symmetric stationary electrovacuum metrics [7].

One is then tempted to speculate that a similar mechanism is working in the case of gravitational radiation, i.e. a flow of gravitational 'energy' would produce vorticity of congruence of observers. For testing such a conjecture we have a tensor quantity available which allows us to define a covariant super-energy density and a super-Poynting vector, namely the Bel–Robinson tensor [8].

When the super-Poynting vector (P^μ), based on the Bel–Robinson tensor (as defined in [9]) is calculated for the Bondi metric [10], we obtain that the contravariant azimuthal component (P^ϕ) of such vector vanishes [3], as expected from the reflection symmetry of the Bondi metric, which, intuitively, seems to be incompatible with the presence of a circular flow of energy in the ϕ direction. However, the vorticity vector, which is orthogonal to the plane of rotation, has in the Bondi spacetime only one non-vanishing contravariant component (ϕ), thereby implying that the plane of the associated rotation is orthogonal to the ϕ direction. Therefore, it is not the ϕ component of P^μ (as mistakenly stated in [3]), the possible source of vorticity, but the θ component, the cause for such an effect, in the Bondi case.

In order to further strengthen the case for the super-Poynting vector as the cause of the mentioned vorticity, we shall consider here the general radiative metric without axial and reflection symmetry [11, 12]. In the latter case we shall obtain a non-vanishing P^ϕ, which we will identify as the cause of the θ component of the vorticity vector. The discussion on these results is presented in the last section. However, before considering the general (Sachs) case it is instructive to review very briefly the basic ideas of the Bondi formalism in the reflection symmetric case.

2. The Bondi formalism

The general form of an axially and reflection-symmetric asymptotically-flat metric given by Bondi is [10]

$$ds^2 = \left(\frac{V}{r}\,e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta}\,du\,dr + 2Ur^2 e^{2\gamma}\,du\,d\theta - r^2(e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\sin^2\theta\,d\phi^2),$$

(1)

where V, β, U and γ are functions of u, r and θ.

We number the coordinates $x^{0,1,2,3} = u, r, \theta, \phi$, respectively. u is a time-like coordinate such that $u = $ constant defines a null surface. In a flat spacetime, this surface coincides with the null light cone open to the future. r is a null coordinate ($g_{rr} = 0$) and θ and ϕ are two angle coordinates (see [10] for details).

The four metric functions are assumed to be expanded in series of $1/r$; then using the field equations, Bondi gets

$$\gamma = cr^{-1} + \left(C - \tfrac{1}{6}c^3\right)r^{-3} + \cdots \tag{2}$$

$$U = -(c_\theta + 2c\cot\theta)r^{-2} + [2N + 3cc_\theta + 4c^2\cot\theta]r^{-3}\cdots \tag{3}$$

$$V = r - 2M - \left(N_\theta + N\cot\theta - c_\theta^2 - 4cc_\theta\cot\theta - \tfrac{1}{2}c^2(1 + 8\cot^2\theta)\right)r^{-1} + \cdots \tag{4}$$

$$\beta = -\tfrac{1}{4}c^2 r^{-2} + \cdots, \tag{5}$$

where c, C, N and M are functions of u and θ; the letters as subscripts denote derivatives.

Bondi shows that field equations allow us to determine the u-derivatives of all functions in β and γ with the exceptions of news functions c_u.

Thus, the main conclusions emerging from Bondi's approach may be summarized as follows.

If γ, M and N are known for some $u = a$ (constant) and c_u (the news function) is known for all u in the interval $a \leqslant u \leqslant b$, then the system is fully determined in that interval. In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting c_u and vice versa. In light of this comment, the relationship between the news function and the occurrence of radiation becomes clear.

Now, for an observer at rest in the frame of (1), the 4-velocity vector has components $u^\alpha = A^{-1}\delta^\alpha_u$, with

$$A \equiv \left(\frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right)^{1/2}. \tag{6}$$

Then, it can be shown that for such an observer, the vorticity vector

$$\omega^\alpha = \frac{1}{2\sqrt{-g}} \eta^{\alpha\eta\iota\lambda} u_\eta u_{\iota,\lambda}, \qquad \eta_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}, \tag{7}$$

where $\eta_{\alpha\beta\gamma\delta} = +1$ for α, β, γ, δ in even order, -1 for α, β, γ, δ in odd order and 0 otherwise, may be written as (see [1] for details)

$$\omega^\alpha = (0, 0, 0, \omega^\phi), \tag{8}$$

giving for the absolute value of ω^α (keeping only the leading term)

$$\Omega = -\frac{1}{2r}(c_{u\theta} + 2c_u \cot\theta). \tag{9}$$

Therefore, up to order $1/r$, a gyroscope at rest in (1) will precess as long as the system radiates ($c_u \neq 0$).

Next, the super-Poynting vector based on the Bel–Robinson tensor, as defined in [9], is

$$P_\alpha = \eta_{\alpha\beta\gamma\delta} E^\beta_\rho H^{\gamma\rho} u^\delta, \tag{10}$$

where $E_{\mu\nu}$ and $H_{\mu\nu}$ are the electric and magnetic parts of the Weyl tensor, respectively, formed from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and its dual $\tilde{C}_{\alpha\beta\gamma\delta}$ by contraction with the 4-velocity vector given by

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta \tag{11}$$

$$H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \tfrac{1}{2}\eta_{\alpha\gamma\epsilon\delta} C^{\epsilon\delta}{}_{\beta\rho} u^\gamma u^\rho. \tag{12}$$

The electric and magnetic parts of the Weyl tensor have been explicitly calculated for the Bondi metric in [3], using those expressions the reader can easily verify that in this case $P^\phi = 0$. As mentioned before, this is to be expected due to the reflection symmetry of the metric.

Indeed, rotation singles out a specific direction of time, which is at variance with the equivalence of both directions of time, implicit in reflection symmetry. For this reason the Bondi metric in the time-independent limit becomes the static Weyl metric and not the general stationary spacetime.

On the other hand, since the associated rotation in this case takes place on the plane orthogonal to the ϕ direction, P^θ is the quantity to be associated with vorticity. Indeed, the vorticity vector describes the rate of rotation, on its orthogonal plane, with respect to the proper time of the set of neighbouring particles, relative to the local compass of inertia [14]. As can be seen from expressions (29)–(31), in the Bondi limit, such a component does not vanish.

Let us now analyse the general case.

3. The general radiative metric

Shortly after the publication of the seminal paper by Bondi *et al* [10], Sachs [11] presented a generalization of the Bondi formalism, relaxing the conditions of axial and reflection symmetry. In this case, the line element reads (we have found it more convenient to follow the notation given in [12, 13] which is slightly different from the original Sachs paper)

$$
\begin{aligned}
ds^2 = (Vr^{-1}\,e^{2\beta} - r^2e^{2\gamma}U^2\cosh 2\delta - r^2e^{-2\gamma}W^2 \\
\times \cosh 2\delta - 2r^2UW\sinh 2\delta)\,du^2 + 2e^{2\beta}\,du\,dr + 2r^2 \\
\times (e^{2\gamma}U\cosh 2\delta + W\sinh 2\delta)\,du\,d\theta \\
+ 2r^2(e^{-2\gamma}W\cosh 2\delta + U\sinh 2\delta)\sin\theta\,du\,d\phi \\
- r^2(e^{2\gamma}\cosh 2\delta\,d\theta^2 + e^{-2\gamma}\cosh 2\delta\sin^2\theta\,d\phi^2 + 2\sinh 2\delta\sin\theta\,d\theta\,d\phi),
\end{aligned} \tag{13}
$$

where $\beta, \gamma, \delta, U, W, V$ are functions of $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi$. Observe that, unlike [13], we adopt the signature -2 as in [12].

The general analysis of the field equations is similar to the one in [10], but of course expressions are far more complicated (see [11–13] for details). In particular, there are now two news functions.

The asymptotic expansion of metric functions read in this case (see [12] for details)

$$
\gamma = cr^{-1} + [C - c^3/6 - 3cd^2/2]r^{-3} + \cdots \tag{14}
$$

$$
\delta = dr^{-1} + [H + c^2d/2 - d^3/6]r^{-3} + \cdots \tag{15}
$$

$$
\begin{aligned}
U = -(c_\theta + 2c\cot\theta + d_\phi\csc\theta)r^{-2} + [2N + 3(cc_\theta + dd_\theta) + 4(c^2 + d^3)\cot\theta \\
- 2(c_\phi d - cd_\phi)\csc\theta]r^{-3} + \cdots
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
W = -(d_\theta + 2d\cot\theta - c_\phi\csc\theta)r^{-2} \\
+ [2Q + 2(c_\theta d - cd_\theta) + 3(cc_\phi + dd_\phi)\csc\theta]r^{-3} + \cdots
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
V = r - 2M - [N_\theta + \cot\theta + Q_\phi\csc\theta - (c^2 + d^2)/2 \\
- (c_\theta^2 + d_\theta^2) - 4(cc_\theta + dd_\theta)\cot\theta - 4(c^2 + d^2)\cot^2\theta \\
- (c_\phi^2 + d_\phi^2)\csc^2\theta + 4(c_\phi d - cd_\phi)\csc\theta\cot\theta \\
+ 2(c_\phi d_\theta - c_\theta d_\phi)\csc\theta]r^{-1} + \cdots
\end{aligned} \tag{18}
$$

$$
\beta = -(c^2 + d^2)r^{-2}/4 + \cdots, \tag{19}
$$

where c, C, d, H, N, Q and M are now functions of u, θ and ϕ. It can be shown [11–13] that field equations allow us to determine the u-derivatives of all functions in β and γ with the exceptions of news functions c_u and d_u, which remain arbitrary and whose existence represents a clear-cut criterium for the presence of gravitational radiation.

Let us first calculate the vorticity for the congruence of observers at rest in (13), whose 4-velocity vector is given by $u^\alpha = A^{-1}\delta_u^\alpha$, where now A is given by

$$
A = (Vr^{-1}\,e^{2\beta} - r^2e^{2\gamma}U^2\cosh 2\delta - r^2e^{-2\gamma}W^2\cosh 2\delta - 2r^2UW\sinh 2\delta)^{1/2}. \tag{20}
$$

Thus, (7) lead us to

$$
\omega^\alpha = (\omega^u, \omega^r, \omega^\theta, \omega^\phi), \tag{21}
$$

where

$$
\omega^u = -\frac{1}{2A^2\sin\theta}\{r^2e^{-2\beta}(WU_r - UW_r) + [2r^2\sinh 2\delta\cosh 2\delta(U^2e^{2\gamma} + W^2e^{-2\gamma})
$$

$$+ 4UWr^2 \cosh^2 2\delta]e^{-2\beta}\gamma_r + 2r^2 e^{-2\beta}(W^2 e^{-2\gamma} - U^2 e^{2\gamma})\delta_r$$
$$+ e^{2\beta}[e^{-2\beta}(U \sinh 2\delta + e^{-2\gamma} W \cosh 2\delta)]_\theta$$
$$- e^{2\beta}[e^{-2\beta}(W \sinh 2\delta + e^{-2\gamma} U \cosh 2\delta]_\phi\} \tag{22}$$

$$\omega^r = \frac{1}{e^{2\beta} \sin\theta}\left\{2r^2 A^{-2}\left[((U^2 e^{2\gamma} + W^2 e^{-2\gamma})\sinh 2\delta \cosh 2\delta\right.\right.$$
$$\left. + UW \cosh^2 2\delta)\gamma_u) + (W^2 e^{-2\gamma} - U^2 e^{2\gamma})\delta_u + \frac{1}{2}(WU_u - UW_u)\right]$$
$$+ A^2[A^{-2}(W e^{-2\gamma}\cosh 2\delta + U \sinh 2\delta)]_\theta$$
$$\left. - A^2[A^{-2}(W \sinh 2\delta + U e^{2\gamma}\cosh 2\delta)]_\phi\right\} \tag{23}$$

$$\omega^\theta = \frac{1}{2r^2 \sin\theta}\{A^2 e^{-2\beta}[r^2 A^{-2}(U \sinh 2\delta + W e^{-2\gamma}\cosh 2\delta)]_r$$
$$- e^{2\beta}A^{-2}[e^{-2\beta}r^2(U \sinh 2\delta + e^{-2\gamma}W \cosh 2\delta)]_u + e^{2\beta}A^{-2}(e^{-2\beta}A^2)_\phi\}, \tag{24}$$

and

$$\omega^\phi = \frac{1}{2r^2 \sin\theta}\{A^2 e^{-2\beta}[r^2 A^{-2}(W \sinh 2\delta + U e^{2\gamma}\cosh 2\delta)]_r$$
$$- e^{2\beta}A^{-2}[r^2 e^{-2\beta}(W \sinh 2\delta + U e^{2\gamma}\cosh 2\delta)]_u + A^{-2} e^{2\beta}(A^2 e^{-2\beta})_\theta\}. \tag{25}$$

Thus, for the leading term of the absolute value of ω^μ we get

$$\Omega = -\frac{1}{2r}[(c_{\theta u} + 2c_u \cot\theta + d_{\phi u}\csc\theta)^2 + (d_{\theta u} + 2d_u \cot\theta - c_{\phi u}\csc\theta)^2]^{1/2}, \tag{26}$$

which of course reduces to (9) in the Bondi (axially and reflection symmetric) case $(d = c_\phi = 0)$.

Next, calculation of the super-Poynting gives the following result:

$$P_\mu = (0, P_r, P_\theta, P_\phi), \tag{27}$$

where the explicit terms are too long to be written at this point.

Although algebraic manipulation by hand is feasible for the Bondi metric, for the Bondi–Sachs one it is quite cumbersome. Thus, we calculated the super-Poynting components using two different sets of Maple scripts (available upon request). One uses the Maple intrinsic procedures to deal with tensors; the electric and magnetic parts of the Weyl tensor were not manipulated explicitly up to the leading term output for the super-Poynting vector components. The most important feature for this procedure, which leads to considerable simplification, is the use of the shift vector $U^A = (U, W/\sin\theta)$ and the 2-surfaces metric of constant u

$$h_{AB} = \begin{pmatrix} e^{2\gamma}\cosh 2\delta & \sinh 2\delta \sin\theta \\ \sinh 2\delta \sin\theta & e^{-2\gamma}\cosh 2\delta \sin^2\theta \end{pmatrix}, \tag{28}$$

where A, B runs from 2 to 3. In our calculations we keep these auxiliary variables, U^A and h_{AB}, as far as possible. The other set of Maple scripts uses the GRTensor II computer algebra package (running on Maple); in this case, all the relevant objects (electric and magnetic parts of the Weyl tensor, super-Poynting vector P^α) were calculated exactly for the metric (13) performing next a series expansion (at $r \to \infty$) for the components of P^α, taking into account expressions (14)–(19) for the metric functions and keeping just the leading term in the series. Both approaches led us to the same results.

The leading terms for each super-Poynting component are

$$P_r = -2r^{-2}(d_{uu}^2 + c_{uu}^2), \tag{29}$$

$$P_\theta = -\frac{2}{r^2 \sin\theta} \left\{ \left[2\left(d_{uu}^2 + c_{uu}^2\right)c + c_{uu}c_u + d_{uu}d_u \right] \cos\theta + \left[c_{uu}c_{\theta u} + d_{uu}d_{\theta u} + \left(c_{uu}^2 + d_{uu}^2\right)c_\theta \right] \sin\theta \right.$$
$$\left. + c_{uu}d_{\phi u} - d_{uu}c_{\phi u} + \left(d_{uu}^2 + c_{uu}^2\right)d_\phi \right\}, \tag{30}$$

$$P_\phi = \frac{2}{r^2} \left\{ 2\left[c_{uu}d_u - d_{uu}c_u - \left(d_{uu}^2 + c_{uu}^2\right)d \right] \cos\theta + \left[c_{uu}d_{\theta u} - d_{uu}c_{\theta u} - \left(c_{uu}^2 + d_{uu}^2\right)d_\theta \right] \sin\theta \right.$$
$$\left. + \left(d_{uu}^2 + c_{uu}^2\right)c_\phi - \left(c_{uu}c_{\phi u} + d_{uu}d_{\phi u}\right) \right\}, \tag{31}$$

from which it follows

$$P^\phi = -\frac{2}{r^4 \sin^2\theta} \left\{ \sin\theta \left[d_{u\theta}c_{uu} - d_{uu}c_{u\theta} \right] + 2\cos\theta \left[c_{uu}d_u - d_{uu}c_u \right] - \left[c_{uu}c_{u\phi} + d_{uu}d_{u\phi} \right] \right\}, \tag{32}$$

which of course vanishes in the Bondi case.

4. Discussion

We have seen so far that gravitational radiation produces vorticity in the congruence of observers at rest in the frame of (13). We conjecture that such vorticity is caused by the presence of a flow of super-energy, as described by the super-Poynting vector (10).

In the case of axial and reflection symmetry (Bondi), the plane of rotation is orthogonal to the unit vector $\hat{e}_\phi$, in the ϕ direction, and accordingly it is the P^θ component which is responsible for such vorticity, whereas the azimuthal component P^ϕ vanishes, in agreement with the fact that the symmetry of the Bondi metric excludes rotation along the ϕ direction.

In the general case, we have shown that the vorticity vector also has components along $\hat{e}_r$ and $\hat{e}_\theta$, thereby implying rotations on the corresponding orthogonal planes. In particular, rotations in the ϕ direction are now allowed ($\omega^\theta \neq 0$), and therefore we should expect a non-vanishing P^ϕ, which is in fact what happens.

Thus, we have shown that there is always a non-vanishing component of P^μ, on the plane orthogonal to a unit vector along which there is a non-vanishing component of vorticity. Inversely, P^μ vanishes on a plane orthogonal to a unit vector along which the component of vorticity vector vanishes (Bondi). This further supports our conjecture about the link between the super-Poynting vector and vorticity.

References

[1] Herrera L and Hernández-Pastora J L 2000 *Class. Quantum Grav.* **17** 3617
[2] Valiente J 2001 *Class. Quantum Grav.* **18** 4311
[3] Herrera L, Santos N O and Carot J 2006 *J. Math. Phys.* **47** 052502
[4] Sorge F, Bini D and de Felice F 2001 *Class. Quantum Grav.* **18** 2945
[5] Bonnor W B 1991 *Phys. Lett.* A **158** 23

[6] Feynman R P, Leighton R B and Sand M 1964 *Lectures on Physics* vol 2 (Reading, MA: Addison–Wesley) pp 27, 28
[7] Herrera L, González G A, Pachón L A and Rueda J A 2006 *Class. Quantum Grav.* **23** 2395
[8] Bel L 1958 *C. R. Acad. Sci.* **247** 1094
    Bel L 1962 *Cah. de Phys.* **16** 59
    Bel L 2000 *Gen. Rel. Grav.* **32** 2047
[9] Maartens R and Basset B A 1998 *Class. Quantum Grav.* **15** 705
[10] Bondi H, van der Burg M G J and Metzner A W K 1962 *Proc. R. Soc.* A **269** 21
[11] Sachs R 1962 *Proc. R. Soc.* A **270** 103
[12] van der Burg M G J 1966 *Proc. R. Soc.* A **294** 112
[13] D'Inverno R A and Vickers J A 1997 *Phys. Rev.* D **56** 772
[14] Rindler W and Perlick V 1999 *Gen. Rel. Grav.* **22** 1067