Relativistic gravitational collapse in noncomoving coordinates: The post-quasistatic approximation

L. Herrera,[1,*] W. Barreto,[2,†] A. Di Prisco,[1] and N. O. Santos[3,4,5,‡]

[1]*Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela*

[2]*Grupo de Relatividad y Astrofísica Teórica, Departamento de Física, Escuela de Ciencias, Núcleo de Sucre, Universidad de Oriente, Cumaná, Venezuela*

[3]*LRM-CNRS/UMR 8540, Université Pierre et Marie Curie, ERGA, Boıte 142, 4 place Jussieu, 75005 Paris Cedex 05, France*

[4]*Laboratório Nacional de Computação Científica, 25651-070 Petrópolis RJ, Brazil*

[5]*Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro RJ, Brazil*

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A general iterative method for the description of evolving self-gravitating relativistic spheres is presented. Modeling is achieved by the introduction of an ansatz whose rationale becomes intelligible and finds full justification within the context of a suitable definition of the post-quasistatic approximation. As examples of the application of the method we discuss three models in the adiabatic case.

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I. INTRODUCTION

The problem of general relativistic gravitational collapse has attracted the attention of researchers since the seminal paper by Oppenheimer and Snyder [1]. The motivation for such interest is easily understood: the gravitational collapse of massive stars represents one of the few observable phenomena where general relativity is expected to play a relevant role. Ever since that work, much has been written by researchers trying to provide models of evolving gravitating spheres. However, this endeavor proved to be difficult and uncertain. Different kinds of advantages and obstacles appear, depending on the approach adopted for the modeling.

Thus, numerical methods (see [2] and references therein) enable researchers to investigate systems that are extremely difficult to handle analytically. In the case of general relativity, numerical models have proved valuable for investigations of strong field scenarios and have been crucial in revealing unexpected phenomena [3]. Even specific difficulties associated with numerical solutions of partial differential equations in the presence of shocks are being overcome [4]. These days, what seems to be the main limitation for numerical relativity is the computational demands for 3D evolution, prohibitive in some cases [5]. Nevertheless, purely numerical solutions usually hinder the investigation of the general, qualitative aspects of the process. On the other hand, analytical solutions although more suitable for a general discussion (see [6] and references therein), are found either for too simplistic equations of state and/or under additional heuristic assumptions whose justification is usually uncertain. Therefore it seems useful to consider nonstatic models which are relatively simple to analyze but still contain some of the essential features of a realistic situation.

Accordingly, it is our purpose in this work to present an approach for modeling the evolution of self-gravitating spheres, that may be regarded as a compromise between the two approaches mentioned above (analytical and numerical). Indeed, the proposed method, starting from any interior (analytical) static spherically symmetric (''seed'') solution to the Einstein equations, leads to a system of ordinary differential equations for quantities evaluated at the boundary surface of the fluid distribution, whose solution (numerical) allows for modeling the dynamics of self-gravitating spheres whose static limit is the original ''seed'' solution.

The approach is based on the introduction of a set of conveniently defined ''effective'' variables, which are the effective pressure and energy density, and a heuristic ansatz on the latter [7], whose rationale and justification become intelligible within the context of the post-quasistatic appproximation defined below. In the quasistatic approximation (see the next section), the effective variables coincide with the corresponding physical variables (pressure and density) and therefore the method may be regarded as an iterative method with each consecutive step corresponding to a stronger departure from equilibrium. In this work, we shall restrict ourselves to the post-quasistatic level (see the next section for details).

At this point it is important to stress a crucial difference between this method and the one proposed many years ago with a similar structure, but based on radiative Bondi coordinates (see [8] and references therein): in the latter the effective variables introduced do not coincide with the corresponding physical variables in the quasistatic approximation (they do coincide in the static limit), and accordingly the ansatz on those variables remains as a heuristic assumption, only justified by the eventual suitability of the models obtained.

The fluid distribution under consideration will be assumed to be dissipative. Indeed, dissipation due to the emission of massless particles (photons and/or neutrinos) is a characteristic process in the evolution of massive stars. In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole, is neutrino emission [9]. Conse-

*Mailing address: Apartado 80793, Caracas 1080A, Venezuela. Email address: laherrera@telcel.net.ve

†Email address: wbarreto@sucre.udo.edu.ve

‡Email address: nos@cbpf.br; santos@ccr.jussieu.fr

quently, in this paper, the matter distribution forming the self-gravitating object will be described as a dissipative fluid.

In the diffusion approximation, it is assumed that the energy flux of radiation (like that of thermal conduction) is proportional to the gradient of temperature. This assumption is in general very sensible, since the mean free path of particles responsible for the propagation of energy in stellar interiors is in general very small as compared with the typical length of the object. Thus, for a main sequence star such as the sun, the mean free path of photons at the center, is of the order of 2 cm. Also, the mean free path of trapped neutrinos in compact cores of densities about 10^{12} g cm^{-3} becomes smaller than the size of the stellar core [10,11]. Furthermore, the observational data collected from supernova 1987A indicates that the regime of radiation transport prevailing during the emission process is closer to the diffusion approximation than to the streaming out limit [12].

However, in many other circumstances, the mean free path of particles transporting energy may be large enough to justify the free streaming approximation. Therefore our formalism will include simultaneously both limiting cases of radiative transport (diffusion and streaming out), allowing for description of a wide range of situations.

In addition to the usual physical variables (energy density, pressure, velocity, heat flow, etc.) we shall also incorporate into our discussion other quantities which are expected to play an important role in the evolution of evolving self-gravitating systems, such as the Weyl tensor, the shear of the fluid, and the Tolman mass. Therefore these quantities will be calculated and used in the process of modeling. It is also worth mentioning that, although the most common method of solving Einstein's equations is to use comoving coordinates (e.g., [13,6]), we shall use noncomoving coordinates, which implies that the velocity of any fluid element (defined with respect to a conveniently chosen set of observers) has to be considered as a relevant physical variable [14].

The plan of the paper is as follows. In Sec. II we define the conventions and give the field equations and expressions for the kinematical and physical variables we shall use, in noncomoving coordinates. The proposed approach is presented and explained in Sec. III. In Sec. IV we illustrate the method by means of three examples. Finally, a discussion of results is presented in Sec. V.

II. RELEVANT EQUATIONS AND CONVENTIONS

A. The field equations

We consider spherically symmetric distributions of a collapsing fluid, which for the sake of completeness we assume to be locally anisotropic, undergoing dissipation in the form of heat flow and/or free streaming radiation, bounded by a spherical surface Σ.

The line element is given in Schwarzschild-like coordinates by

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{1}$$

where $\nu(t,r)$ and $\lambda(t,r)$ are functions of their arguments. We number the coordinates $x^0=t;\quad x^1=r;\quad x^2=\theta;\quad x^3=\phi$.

The metric (1) has to satisfy the Einstein field equations

$$G^{\nu}_{\mu}=-8\pi T^{\nu}_{\mu}, \tag{2}$$

which in our case read [15]

$$-8\pi T^0_0=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right), \tag{3}$$

$$-8\pi T^1_1=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right), \tag{4}$$

$$\begin{aligned}-8\pi T^2_2&=-8\pi T^3_3\\ &=-\frac{e^{-\nu}}{4}[2\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})]\\ &\quad+\frac{e^{-\lambda}}{4}\left(2\nu''+\nu'^2-\lambda'\nu'+2\frac{\nu'-\lambda'}{r}\right),\end{aligned} \tag{5}$$

$$-8\pi T_{01}=-\frac{\dot{\lambda}}{r}, \tag{6}$$

where overdots and primes stand for partial differentiation with respect to t and r, respectively.

In order to give physical significance to the T^{μ}_{ν} components we apply the Bondi approach [15].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates (τ,x,y,z)

$$d\tau=e^{\nu/2}dt;\quad dx=e^{\lambda/2}dr;\quad dy=rd\theta;\quad dz=r\sin\theta d\phi.$$

Then, denoting the Minkowski components of the energy tensor by an overbar, we have

$$\overline{T^0_0}=T^0_0,\quad \overline{T^1_1}=T^1_1,\quad \overline{T^2_2}=T^2_2,\quad \overline{T^3_3}=T^3_3,$$

$$\overline{T_{01}}=e^{-(\nu+\lambda)/2}T_{01}.$$

Next, we suppose that, when viewed by an observer moving relative to these coordinates with proper velocity ω in the radial direction, the physical content of space consists of an anisotropic fluid of energy density ρ, radial pressure P_r, tangential pressure $P_\perp$, radial heat flux $\hat{q}$, and unpolarized radiation of energy density $\hat{\epsilon}$ traveling in the radial direction. Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

$$\begin{pmatrix}\rho+\hat{\epsilon} & -\hat{q}-\hat{\epsilon} & 0 & 0\\ -\hat{q}-\hat{\epsilon} & P_r+\hat{\epsilon} & 0 & 0\\ 0 & 0 & P_\perp & 0\\ 0 & 0 & 0 & P_\perp\end{pmatrix}.$$

Then a Lorentz transformation readily shows that

$$T^0_0=\overline{T^0_0}=\frac{\rho+P_r\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}+\epsilon, \tag{7}$$

$$T_1^1=\overline{T}_1^1=-\frac{P_r+\rho\omega^2}{1-\omega^2}-\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}-\epsilon, \tag{8}$$

$$T_2^2=T_3^3=\overline{T}_2^2=\overline{T}_3^3=-P_\perp, \tag{9}$$

$$\begin{aligned}T_{01}&=e^{(\nu+\lambda)/2}\overline{T}_{01}\\&=-\frac{(\rho+P_r)\omega e^{(\nu+\lambda)/2}}{1-\omega^2}\\&\quad-\frac{Qe^{\nu/2}e^{\lambda}}{(1-\omega^2)^{1/2}}(1+\omega^2)-e^{(\nu+\lambda)/2}\epsilon,\end{aligned} \tag{10}$$

with

$$Q\equiv\frac{\hat{q}e^{-\lambda/2}}{(1-\omega^2)^{1/2}} \tag{11}$$

and

$$\epsilon\equiv\hat{\epsilon}\frac{(1+\omega)}{(1-\omega)}. \tag{12}$$

Note that the coordinate velocity in the (t,r,θ,ϕ) system, dr/dt, is related to ω by

$$\omega=\frac{dr}{dt}e^{(\lambda-\nu)/2}. \tag{13}$$

Feeding back Eqs. (7)–(10) into Eqs. (3)–(6), we get the field equations in the form

$$\begin{aligned}&\frac{\rho+P_r\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}+\epsilon\\&\quad=-\frac{1}{8\pi}\left\{-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}&\frac{P_r+\rho\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}+\epsilon\\&\quad=-\frac{1}{8\pi}\left\{\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)\right\},\end{aligned} \tag{15}$$

$$\begin{aligned}P_\perp=-\frac{1}{8\pi}\Bigg\{&\frac{e^{-\nu}}{4}[2\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})]\\&-\frac{e^{-\lambda}}{4}\left(2\nu''+\nu'^2-\lambda'\nu'+2\frac{\nu'-\lambda'}{r}\right)\Bigg\},\end{aligned} \tag{16}$$

$$\begin{aligned}&\frac{(\rho+P_r)\omega e^{(\nu+\lambda)/2}}{1-\omega^2}+\frac{Qe^{\nu/2}e^{\lambda}}{(1-\omega^2)^{1/2}}(1+\omega^2)\\&\quad+e^{(\nu+\lambda)/2}\epsilon=-\frac{\dot{\lambda}}{8\pi r}.\end{aligned} \tag{17}$$

Observe that if ν and λ are fully specified then Eqs. (14)–(17) become a system of algebraic equations for the physical variables ρ, P_r, $P_\perp$, ω, Q, and ϵ. Obviously, in the most general case when all these variables are nonvanishing, the system is underdetermined, and two equations of state should be given. In general, whenever $Q\neq 0$ a transport equation has to be assumed. In the case originally considered by Bondi [15] (locally isotropic fluid and free streaming regime, $Q=0$) the system is closed. For the adiabatic ($\epsilon=Q=0$) and locally isotropic fluid ($P_r=P_\perp$) the system is overdetermined, and a constraint on the physical variables appears.

At the outside of the fluid distribution, the spacetime is that of Vaidya, given by

$$\begin{aligned}ds^2=&\left(1-\frac{2M(u)}{\mathcal{R}}\right)du^2+2dud\mathcal{R}\\&-\mathcal{R}^2(d\theta^2+\sin^2\theta d\phi^2),\end{aligned} \tag{18}$$

where u is a coordinate related to the retarded time, such that $u=$const is (asymptotically) a null cone open to the future and $\mathcal{R}$ is a null coordinate ($g_{\mathcal{R}\mathcal{R}}=0$). It should be remarked, however, that strictly speaking the radiation can be considered in radial free streaming only at radial infinity.

The two coordinate systems (t,r,θ,ϕ) and $(u,\mathcal{R},\theta,\phi)$ are related at the boundary surface and outside it by

$$u=t-r-2M\ln\left(\frac{r}{2M}-1\right), \tag{19}$$

$$\mathcal{R}=r. \tag{20}$$

In order to smoothly match the two metrics above on the boundary surface $r=r_\Sigma(t)$, we first require the continuity of the first fundamental form across that surface. Then

$$[e^{\nu_\Sigma}-e^{\lambda_\Sigma}\dot{r}_\Sigma^2]\,dt^2=\left[1-\frac{2M}{R_\Sigma}+2\frac{dR_\Sigma}{du}\right]du^2, \tag{21}$$

where $R=R_\Sigma(u)$ is the equation of the boundary surface in (u,R,θ,ϕ) coordinates.

From Eq. (21), using Eqs. (13), (19), and (18), it follows that

$$e^{\nu_\Sigma}=1-\frac{2M}{R_\Sigma}, \tag{22}$$

$$e^{-\lambda_\Sigma}=1-\frac{2M}{R_\Sigma}. \tag{23}$$

where, from now on, the subscript Σ indicates that the quantity is evaluated at the boundary surface Σ.

Next, the unit vector n_μ, normal to the boundary surface, has the components

$$n_\mu^{(+)}=\left(-\beta\frac{dR_\Sigma}{du},\beta,0,0\right), \tag{24}$$

where $+$ indicates that the components are evaluated from the outside of Σ, and β is given by

$$\beta=\left(1-\frac{2M(u)}{R_\Sigma}+2\frac{dR_\Sigma}{du}\right)^{-1/2}. \tag{25}$$

The unit vector normal to Σ, evaluated from the inside, is given by

$$n_\mu^{(-)}=(-\dot{r}_\Sigma\gamma,\gamma,0,0) \tag{26}$$

with

$$\gamma=(e^{-\lambda_\Sigma}-\dot{r}_\Sigma^2 e^{-\nu_\Sigma})^{-1/2}. \tag{27}$$

Let us now define a timelike vector v^μ such that

$$v^{\mu(+)}=\beta\delta_u^\mu+\beta\frac{dR_\Sigma}{du}\delta_R^\mu \tag{28}$$

and

$$v^{\mu(-)}=\frac{e^{-\nu_\Sigma/2}}{(1-\omega_\Sigma^2)^{1/2}}\delta_t^\mu+\frac{\omega_\Sigma e^{-\lambda_\Sigma/2}}{(1-\omega_\Sigma^2)^{1/2}}\delta_r^\mu. \tag{29}$$

Then, junction conditions across Σ, require [in addition to Eq. (21)]

$$(T_{\mu\nu}n^\mu n^\nu)_\Sigma^{(+)}=(T_{\mu\nu}n^\mu n^\nu)_\Sigma^{(-)}, \tag{30}$$

$$(T_{\mu\nu}n^\mu v^\nu)_\Sigma^{(+)}=(T_{\mu\nu}n^\mu v^\nu)_\Sigma^{(-)}, \tag{31}$$

where the expressions for the energy-momentum tensor at both sides of the boundary surface are

$$T_{\mu\nu}^{(-)}=(\rho+P_\perp)u_\mu u_\nu-P_\perp g_{\mu\nu}+(P_r-P_\perp)s_\mu s_\nu +q_\mu u_\nu+q_\nu u_\mu+\epsilon l_\nu l_\mu \tag{32}$$

and

$$T_{\mu\nu}^{(+)}=-\frac{1}{4\pi R^2}\frac{dM}{du}\delta_\mu^0\delta_\nu^0, \tag{33}$$

with

$$u^\mu=\left(\frac{e^{-\nu/2}}{(1-\omega^2)^{1/2}},\quad \frac{\omega e^{-\lambda/2}}{(1-\omega^2)^{1/2}},0,0\right), \tag{34}$$

$$s^\mu=\left(\frac{\omega e^{-\nu/2}}{(1-\omega^2)^{1/2}},\quad \frac{e^{-\lambda/2}}{(1-\omega^2)^{1/2}},0,0\right), \tag{35}$$

$$l^\mu=(e^{-\nu/2},e^{-\lambda/2},0,0), \tag{36}$$

where u^μ denotes the four-velocity of the fluid, s^μ is a radially directed spacelike vector orthogonal to u^μ, l^μ is a null outgoing vector, and

$$q^\mu=Q(\omega e^{(\lambda-\nu)/2},1,0,0). \tag{37}$$

Then it follows from Eqs. (30) and (31) that

$$[P_r+\hat{\epsilon}]_\Sigma=-\left[\frac{1}{4\pi R^2}\frac{dM}{du}\beta^2\right]_\Sigma, \tag{38}$$

$$[Qe^{\lambda/2}(1-\omega^2)^{1/2}+\hat{\epsilon}]_\Sigma=-\left[\frac{1}{4\pi R^2}\frac{dM}{du}\beta^2\right]_\Sigma. \tag{39}$$

Equations (21), (38), and (39) are the necessary and sufficient conditions for a smooth matching of the two metrics (1) and (18) on Σ. Combining Eqs. (38) and (39) we get

$$[P_r]_\Sigma=[Q\,e^{\lambda/2}(1-\omega^2)^{1/2}]_\Sigma, \tag{40}$$

expressing the discontinuity of the radial pressure in the presence of heat flow, which is a well known result [16].

Next, it will be useful to calculate the radial component of the conservation law

$$T_{\nu;\mu}^\mu=0. \tag{41}$$

After tedious but simple calculations we get

$$(-8\pi T_1^1)'=\frac{16\pi}{r}(T_1^1-T_2^2)+4\pi\nu'(T_1^1-T_0^0) +\frac{e^{-\nu}}{r}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\nu}}{2}\right), \tag{42}$$

which in the static case becomes

$$P_r'=-\frac{\nu'}{2}(\rho+P_r)+\frac{2(P_\perp-P_r)}{r}, \tag{43}$$

representing the generalization of the Tolman-Oppenheimer-Volkof equation for anisotropic fluids [17].

B. The kinematical variables

The components of the shear tensor are defined by

$$\sigma_{\mu\nu}=u_{\mu;\nu}+u_{\nu;\mu}-u_\mu a_\nu-u_\nu a_\mu-\frac{2}{3}\Theta P_{\mu\nu}, \tag{44}$$

where

$$P_{\mu\nu}=g_{\mu\nu}-u_\mu u_\nu,\quad \Theta=u^\mu_{;\mu},$$
$$a_\mu=u^\nu u_{\mu;\nu}, \tag{45}$$

denote the projector onto the three-space orthogonal to u^μ, the expansion, and the four-acceleration, respectively. A simple calculation gives

$$\Theta=\frac{e^{-\nu/2}}{2(1-\omega^2)^{1/2}}\left(\dot{\lambda}+\frac{2\omega\dot{\omega}}{1-\omega^2}\right)+\frac{e^{-\lambda/2}}{2(1-\omega^2)^{1/2}} \times\left(\omega\nu'+2\omega'+\frac{2\omega^2\omega'}{1-\omega^2}+\frac{4\omega}{r}\right), \tag{46}$$

$$\sigma_{11}=-\frac{2}{3(1-\omega^2)^{3/2}}\left[e^{\lambda}e^{-\nu/2}\left(\dot{\lambda}+\frac{2\omega\dot{\omega}}{1-\omega^2}\right)\right.$$
$$\left.+e^{\lambda/2}\left(\omega\nu'+\frac{2\omega'}{1-\omega^2}-\frac{2\omega}{r}\right)\right], \tag{47}$$

$$\sigma_{22}=-\frac{e^{-\lambda}r^2(1-\omega^2)}{2}\sigma_{11}, \tag{48}$$

$$\sigma_{33}=-\frac{e^{-\lambda}r^2(1-\omega^2)}{2}\sin^2\theta\sigma_{11}, \tag{49}$$

$$\sigma_{00}=\omega^2e^{-\lambda}e^{\nu}\sigma_{11}, \tag{50}$$

$$\sigma_{01}=-\omega e^{(\nu-\lambda)/2}\sigma_{11}, \tag{51}$$

$$a_0=\frac{1}{1-\omega^2}\left[\left(\frac{\omega\dot{\omega}}{1-\omega^2}+\frac{\omega^2\dot{\lambda}}{2}\right)+e^{\nu/2}e^{-\lambda/2}\left(\frac{\omega\nu'}{2}+\frac{\omega^2\omega'}{1-\omega^2}\right)\right], \tag{52}$$

$$a_1=-\frac{1}{1-\omega^2}\left[\left(\frac{\omega\omega'}{1-\omega^2}+\frac{\nu'}{2}\right)+e^{-\nu/2}e^{\lambda/2}\left(\frac{\omega\dot{\lambda}}{2}+\frac{\dot{\omega}}{1-\omega^2}\right)\right], \tag{53}$$

and, for the shear scalar σ,

$$\sigma=\sqrt{3}\left(\frac{\Theta}{3}-\frac{e^{-\lambda/2}}{r}\frac{\omega}{\sqrt{1-\omega^2}}\right). \tag{54}$$

C. The Tolman mass

The Tolman mass for a spherically symmetric distribution of matter is given by [Eq. (24) in [18]]

$$m_T=4\pi\int_0^{r_\Sigma}r^2e^{(\nu+\lambda)/2}(T_0^0-T_1^1-2T_2^2)dr$$
$$+\frac{1}{2}\int_0^{r_\Sigma}r^2e^{(\nu+\lambda)/2}\frac{\partial}{\partial t}\left(\frac{\partial\pounds}{\partial[\partial(g^{\alpha\beta}\sqrt{-g})/\partial t]}\right)g^{\alpha\beta}dr, \tag{55}$$

where £ denotes the usual gravitational Lagrangian density [Eq. (10) in [18]]. Although Tolman's formula was introduced as a measure of the total energy of the system, with no commitment to its localization, we shall define the mass within a sphere of radius r, completely inside Σ, as

$$m_T=4\pi\int_0^{r}r^2e^{(\nu+\lambda)/2}(T_0^0-T_1^1-2T_2^2)dr$$
$$+\frac{1}{2}\int_0^{r}r^2e^{(\nu+\lambda)/2}\frac{\partial}{\partial t}\left(\frac{\partial\pounds}{\partial[\partial(g^{\alpha\beta}\sqrt{-g})/\partial t]}\right)g^{\alpha\beta}dr. \tag{56}$$

This extension of the global concept of energy to a local level [19] is suggested by the conspicuous role played by m_T as the "effective gravitational mass," which will be exhibited below. Even though Tolman's definition is not without its problems [19,20], we shall see that m_T, as defined by Eq. (56), is a good measure of the active gravitational mass, at least for the systems under consideration.

Let us now evaluate expression (56). The first integral in that expression (I),

$$\mathrm{I}\equiv4\pi\int_0^r r^2e^{(\nu+\lambda)/2}(T_0^0-T_1^1-2T_2^2)dr, \tag{57}$$

may be transformed to give (see [21] for details)

$$\mathrm{I}=e^{(\nu+\lambda)/2}\left[m(r,t)-\frac{4\pi}{3}r^3T_1^1\right]$$
$$-\int_0^r e^{(\lambda-\nu)/2}\frac{r^2}{2}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\nu}}{2}\right)dr, \tag{58}$$

where the mass function m, as usual, is defined by

$$e^{-\lambda(r,t)}=1-2m(r,t)/r. \tag{59}$$

Next, from [Eq. (13) in [18]],

$$\frac{\partial}{\partial t}\left(\frac{\partial\pounds}{\partial[\partial(g^{\alpha\beta}\sqrt{-g})/\partial t]}\right)$$
$$=-\Gamma^0_{\alpha\beta}+\frac{1}{2}\delta^0_\alpha\Gamma^\sigma_{\beta\sigma}+\frac{1}{2}\delta^0_\beta\Gamma^\sigma_{\alpha\sigma}, \tag{60}$$

and so the second integral (II) in Eq. (56) may be expressed as

$$\mathrm{II}=\frac{1}{2}\int_0^r r^2e^{(\lambda-\nu)/2}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\nu}}{2}\right)dr. \tag{61}$$

Thus

$$m_T\equiv\mathrm{I}+\mathrm{II}=e^{(\nu+\lambda)/2}[m(r,t)-4\pi r^3T_1^1]. \tag{62}$$

This is, formally, the same expression for m_T in terms of m and T_1^1 that appears in the static (or quasistatic) case [Eq. (25) in [22]].

Replacing T_1^1 by Eq. (4) and m by Eq. (59), one also finds

$$m_T=e^{(\nu-\lambda)/2}\nu'\frac{r^2}{2}. \tag{63}$$

This last equation brings out the physical meaning of m_T as the active gravitational mass. Indeed, it can be easily shown [23] that the gravitational acceleration a of a test particle, instantaneously at rest in a static gravitational field, as measured with standard rods and a coordinate clock is given by

$$a=-\frac{e^{(\nu-\lambda)/2}\nu'}{2}=-\frac{m_T}{r^2}. \tag{64}$$

A similar conclusion can be obtained by inspection of Eq. (43) (valid only in the static or quasistatic case) [24]. In fact, the first term on the right side of this equation (the "gravitational force" term) is a product of the "passive" gravitational mass density $\rho+P_r$ and a term proportional to m_T/r^2.

D. The Weyl tensor

Since the publication of Penrose's work [25], there has been increasing interest in studying the possible role of the Weyl tensor (or some function of it) in the evolution of self-gravitating systems [26]. This interest is reinforced by the fact that, for a spherically symmetric distribution of fluid, the Weyl tensor may be defined exclusively in terms of the density contrast and the local anisotropy of the pressure (see below), which in turn are known to affect the fate of gravitational collapse [27].

Now, using MAPLE V, it is found that all nonvanishing components of the Weyl tensor are proportional to

$$W \equiv \frac{r}{2}C^3_{232} = W_{(s)} + \frac{r^3 e^{-\nu}}{12}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right) \tag{65}$$

where

$$W_{(s)} = \frac{r^3 e^{-\lambda}}{6}\left(\frac{e^{\lambda}}{r^2} - \frac{1}{r^2} + \frac{\nu'\lambda'}{4} - \frac{\nu'^2}{4} - \frac{\nu''}{2} - \frac{\lambda'}{2r} + \frac{\nu'}{2r}\right) \tag{66}$$

corresponds to the contribution in the static case.

Also, from the field equations and the definition of the Weyl tensor it can be easily shown that (see [21] for details)

$$W = -\frac{4\pi}{3}\int_0^r r^3 (T_0^0)' dr + \frac{4\pi}{3} r^3 (T_2^2 - T_1^1). \tag{67}$$

III. THE METHOD

We now have available all the ingredients required to present our method; however, before doing so some general considerations will be necessary.

A. Equilibrium and departures from equilibrium

The simplest situation, when dealing with self-gravitating spheres, is that of equilibrium (static case). In our notation that means that $\omega=\epsilon=Q=0$, all time derivatives vanishes, and we obtain the generalized Tolman-Oppenheimer-Volkof equation (43).

Next, we have the quasistatic regime. By this we mean that the sphere changes slowly, on a time scale that is very long compared to the typical time in which the sphere reacts to a slight perturbation of hydrostatic equilibrium; this typical time scale is called the hydrostatic time scale [28] (sometimes this time scale is also referred to as the dynamical time scale, e.g., see the third reference in [28]). Thus, in this regime the system is always very close to hydrostatic equilibrium and its evolution may be regarded as a sequence of static models linked by Eq. (17). This assumption is very sensible because the hydrostatic time scale is very small for many phases of the life of the star. It is of the order of 27 min for the Sun, 4.5 s for a white dwarf, and 10^{-4} s for a neutron star of one solar mass and 10 km radius. It is well known that all the stellar configurations mentioned above generally change on time scales that are very long compared to their respective hydrostatic time scales. Let us now translate this assumption into conditions on ω and the metric functions.

First of all, slow contraction means that the radial velocity ω as measured by the Minkowski observer is always much smaller than the velocity of light ($\omega \ll 1$). Therefore we have to neglect terms of order $O(\omega^2)$.

Then Eq. (42) yields

$$\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\nu}\dot{\lambda}}{2} = 8\pi r e^{\nu}\left[(P_r+\epsilon)' + (\rho+P_r+2\epsilon)\frac{\nu'}{2} - 2\frac{P_\perp - P_r - \epsilon}{r}\right] \tag{68}$$

(observe the contribution of ϵ to both P_r and ρ, and the fact that ϵ, ω, and Q are of the same order of smallness, in this approximation).

Since, by assumption, in this regime the system is always (not only at a given time t) in equilibrium (or very close to it), Eqs. (43) and (68) imply that, for an arbitrary slowly evolving configuration,

$$\ddot{\lambda} \approx \dot{\nu}\dot{\lambda} \approx \dot{\lambda}^2 \approx 0, \tag{69}$$

and of course, time derivatives of any order of the left hand side of the hydrostatic equilibrium equation must also vanish, for otherwise the system will deviate from equilibrium. This condition implies, in particular, that we must demand in this regime

$$\ddot{\nu} \approx 0.$$

Finally, from the time derivative of Eq. (6), and using Eq. (10), it follows that

$$\dot{\omega} \approx O(\ddot{\lambda}, \dot{\lambda}\omega, \dot{\nu}\omega), \tag{70}$$

which implies that we also have to neglect terms linear in the acceleration. From purely physical considerations, it is obvious that the vanishing of $\dot{\omega}$ is required to keep the system always in equilibrium.

Thus, in the quasistatic regime we have to assume

$$O(\omega^2) = \dot{\lambda}^2 = \dot{\nu}^2 = \dot{\lambda}\dot{\nu} = \ddot{\lambda} = \ddot{\nu} = 0, \tag{71}$$

implying that the system remains in (or very close to) equilibrium. However, during their evolution, self-gravitating objects may pass through phases of intense dynamical activity, with time scales of the order of magnitude of (or even smaller than) the hydrostatic time scale, and for which the quasistatic approximation is clearly not reliable (e.g., the collapse of very massive stars [29] and the quick collapse phase preceding neutron star formation; see, for example, [30] and references therein). In these cases it is mandatory to take into account terms that describe departure from equilibrium.

B. The effective variables and the post-quasistatic approximation

Let us now define the following effective variables:

$$\tilde{\rho}=T_0^0=\frac{\rho+P_r\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}+\epsilon, \tag{72}$$

$$\tilde{P}=-T_1^1=\frac{P_r+\rho\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\lambda/2}}{(1-\omega^2)^{1/2}}+\epsilon. \tag{73}$$

In the quasistatic regime the effective variables satisfy the same equation (43) as the corresponding physical variables (taking into account the contribution of ϵ to the "total" energy density and radial pressure, whenever the free streaming approximation is being used). Therefore, in the quasistatic situation (and obviously in the static too), effective and physical variables share the same radial dependence. Next, feeding back Eqs. (72) and (73) into Eqs. (14) and (15), these two equations may be formally integrated, to obtain

$$m=4\pi\int_0^r r^2\tilde{\rho}dr, \tag{74}$$

$$\nu=\nu_\Sigma+\int_{r_\Sigma}^r\frac{2(4\pi r^3\tilde{P}+m)}{r(r-2m)}dr. \tag{75}$$

From here it is obvious that for a given radial dependence of the effective variables the radial dependence of the metric functions become completely determined.

With this last comment in mind, we shall define the post-quasistatic regime as that corresponding to a system out of equilibrium (or quasiequilibrium) but whose effective variables share the same radial dependence as the corrresponding physical variables in the state of equilibrium (or quasiequilibrium). Alternatively, it may be said that the system in the post-quasistatic regime is characterized by metric functions whose radial dependence is the same as the metric functions corresponding to the static (quasistatic) regime. The rationale behind this definition is not difficult to grasp: we look for a regime which, although out of equilibrium, represents the closest possible situation to a quasistatic evolution (see more on this point in Sec. V).

C. The algorithm

Let us now outline the approach that we propose.

(1) Take an interior solution to the Einstein equations, representing a fluid distribution of matter in equilibrium, with a given

$$\rho_{st}=\rho(r), \quad P_{rst}=P_r(r).$$

(2) Assume that the r dependence of $\tilde{P}$ and $\tilde{\rho}$ is the same as that of P_{rst} and ρ_{st}, respectively.

(3) Using Eqs. (75) and (74), with the r dependence of $\tilde{P}$ and $\tilde{\rho}$, one gets m and ν up to some functions of t, which will be specified below.

(4) For these functions of t one has three ordinary differential equations (hereafter referred to as surface equations), namely, (a) Eq. (13) evaluated on $r=r_\Sigma$; (b) Eq. (42) evaluated on $r=r_\Sigma$; and (c) the equation relating the total mass loss rate to the energy flux through the boundary surface.

(5) Depending on the kind of matter under consideration, the system of surface equations described above may be closed with the additional information provided by the transport equation and/or the equation of state for the anisotropic pressure and/or additional information about some of the physical variables evaluated on the boundary surface (e.g., the luminosity).

(6) Once the system of surface equations is closed, it may be integrated for any particular initial data.

(7) Feeding back the result of integration into the expressions for m and ν, these two functions are completely determined.

(8) With the input from point 7 above, and using the field equations, together with the equations of state and/or the transport equation, all physical variables may be found for any piece of matter distribution.

D. The surface equations

As should be clear from the above the crucial point in the algorithm is the system of surface equations. So, let us specify them now.

Introducing the dimensionless variables

$$A=r_\Sigma/m_\Sigma(0),$$

$$F=1-2M/A,$$

$$M=m_\Sigma/m_\Sigma(0),$$

$$\Omega=\omega_\Sigma,$$

$$\alpha=t/m_\Sigma(0),$$

where $m_\Sigma(0)$ denotes the total initial mass, we obtain the first surface equation by evaluating Eq. (13) at $r=r_\Sigma$. One gets

$$\frac{dA}{d\alpha}=F\Omega. \tag{76}$$

Next, using junction conditions, one obtains from Eqs. (59), (14), and (17) evaluated at $r=r_\Sigma$ that

$$\frac{dM}{d\alpha}=-F(1+\Omega)\hat{E}, \tag{77}$$

with

$$\hat{E}=4\pi r_\Sigma^2(\hat{\epsilon}_\Sigma+\hat{q}_\Sigma), \tag{78}$$

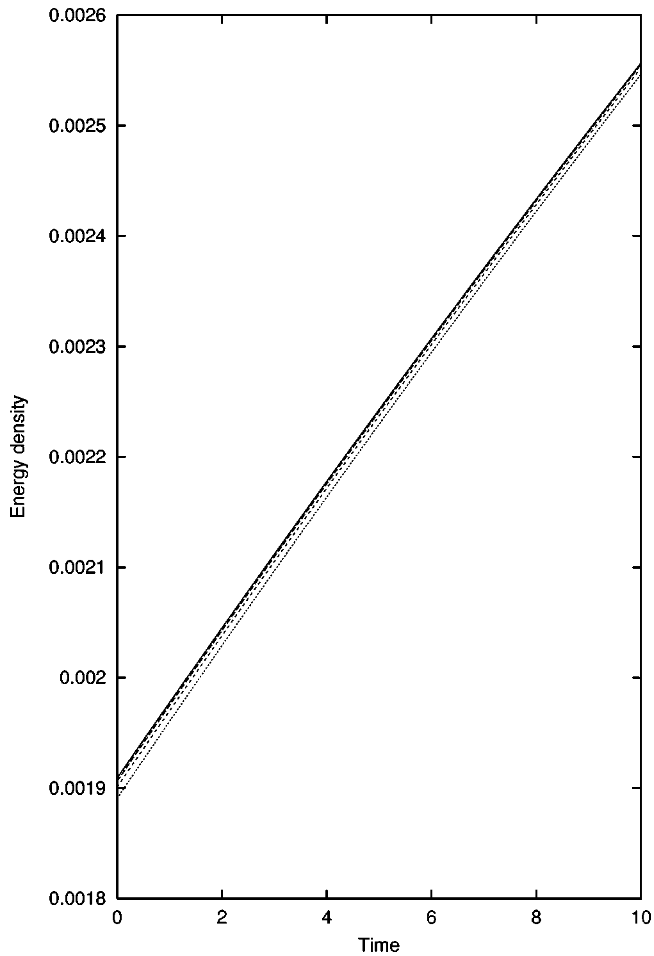

FIG. 1. Energy density $\rho m(0)^2$ as a function of (dimensionless) time (α) for the Schwarzschild-type model. The initial conditions are $A(0)=5$, $F(0)=0.6$, and $\Omega(0)=-0.1$. Curves represent different regions: $r/r_\Sigma=0.25$ (continuous line); 0.50 (dashed line); 0.75 (short-dashed line); and 1.00 (dotted line).

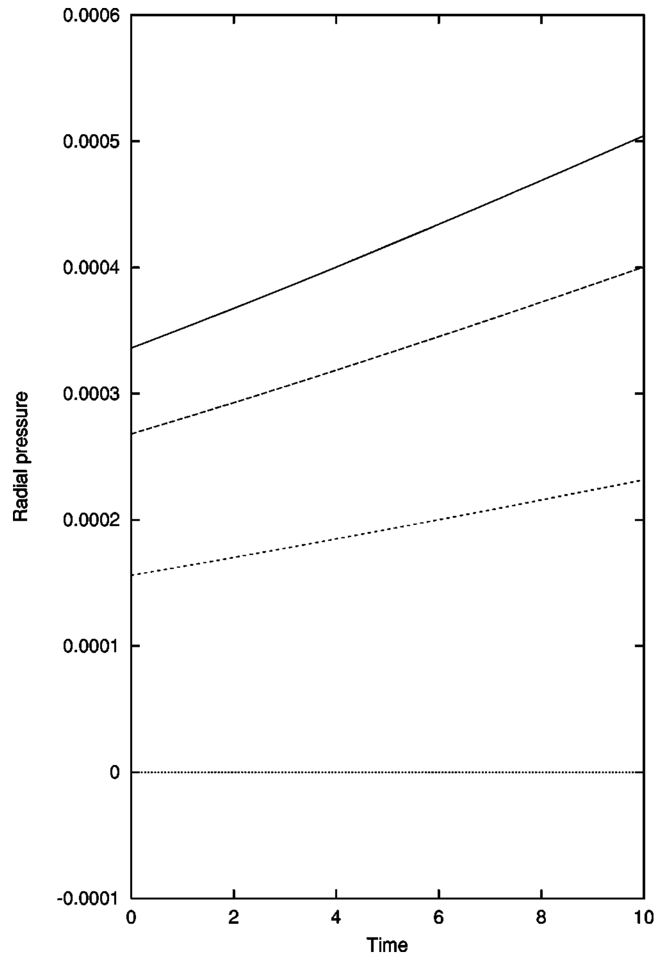

FIG. 2. Radial pressure $P_r m(0)^2$ as a function of time for the Schwarzschild-type model. The initial conditions are $A(0)=5$, $F(0)=0.6$, and $\Omega(0)=-0.1$. Curves represent different regions: $r/r_\Sigma=0.25$ (continuous line); 0.50 (dashed line); 0.75 (short-dashed line); and 1.00 (dotted line).

where the first and second terms on the right of Eq. (77) represent the gravitational redshift and the Doppler shift corrections, respectively. Then, defining the luminosity perceived by an observer at infinity as

$$L=-\frac{dM}{d\alpha},$$

we obtain the second surface equation in the form

$$\frac{dF}{d\alpha}=\frac{F}{A}(1-F)\Omega+2L/A. \tag{79}$$

The third surface equation may be obtained by evaluating at the boundary surface the conservation law $T^{\mu}_{1;\mu}=0$, which reads

$$\tilde{P}'+\frac{(\tilde{\rho}+\tilde{P})(4\pi r^3\tilde{P}+m)}{r(r-2m)} =\frac{e^{-\nu}}{4\pi r(r-2m)}\left(\ddot{m}+\frac{3\dot{m}^2}{r-2m}-\frac{\dot{m}\dot{\nu}}{2}\right)+\frac{2}{r}(P_\perp-\tilde{P}). \tag{80}$$

Now, in the following section we consider two relatively simple models with a separable effective density, i.e., $\tilde{\rho}=f(t)h(r)$; thus Eq. (80) evaluated at the boundary surface leads to

$$\frac{d\Omega}{d\alpha}=\Omega^2\left[\frac{8F}{A}+2Fk(r_\Sigma)+4\pi\tilde{\rho}_\Sigma A(3-\Omega^2)\right] -\frac{F}{\tilde{\rho}_\Sigma}\left[R-\frac{2}{A}\left(P_{\perp\Sigma}-\tilde{\rho}_\Sigma\Omega^2-\frac{\overline{E}(1+\Omega)}{4\pi r_\Sigma^2}\right)\right], \tag{81}$$

where

$$R=\left[\tilde{P}'+\frac{\tilde{P}+\tilde{\rho}}{1-2m/r}\left(4\pi r\tilde{P}+\frac{m}{r^2}\right)\right]_\Sigma, \tag{82}$$

$$\overline{E}=\hat{E}(1+\Omega), \tag{83}$$

and

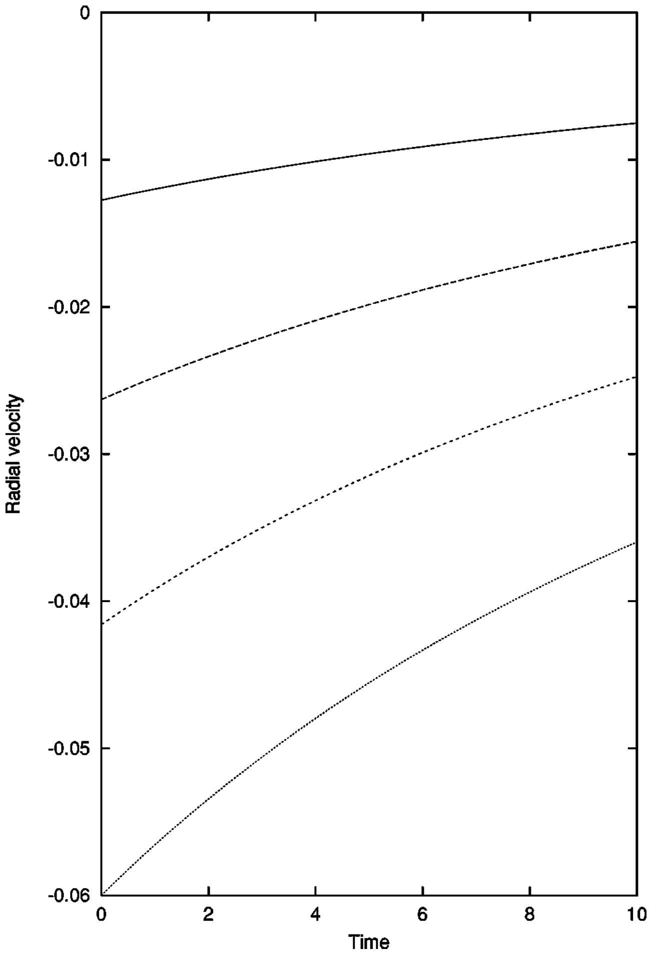

FIG. 3. Radial velocity ω as a function of time for the Schwarzschild-type model. The initial conditions are $A(0)=5$, $F(0)=0.6$, and $\Omega(0)=-0.1$. Curves represent different regions: $r/r_\Sigma=0.25$ (continuous line); 0.50 (dashed line); 0.75 (short-dashed line); and 1.00 (dotted line).

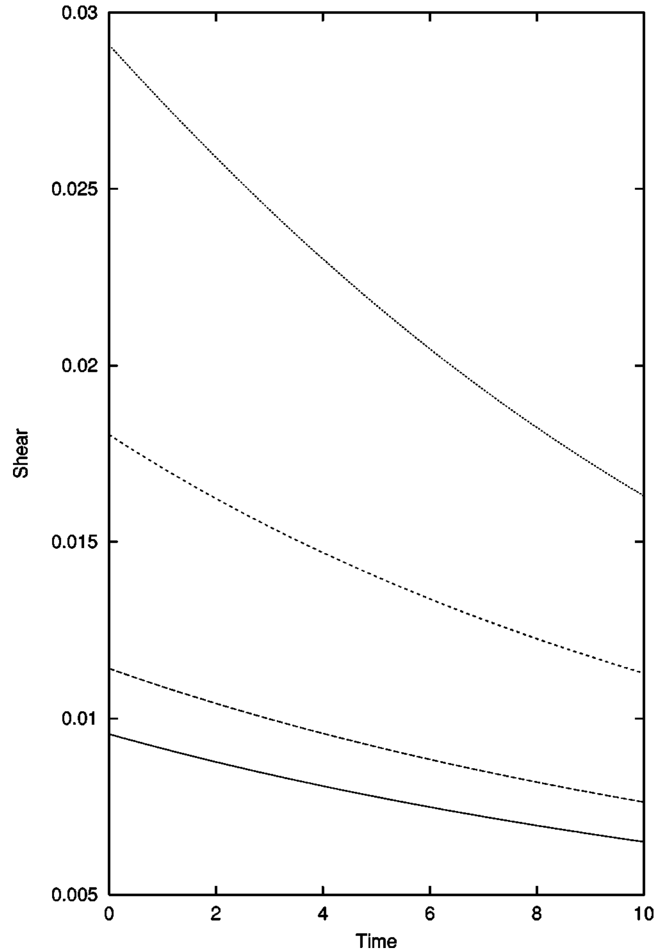

FIG. 4. Shear $\sigma m(0)$ as a function of time for the Schwarzschild-type model. The initial conditions are $A(0)=5$, $F(0)=0.6$, and $\Omega(0)=-0.1$. Curves represent different regions: $r/r_\Sigma=0.25$ (continuous line); 0.50 (dashed line); 0.75 (short-dashed line); and 1.00 (dotted line).

$$k(r_\Sigma)=\frac{d}{dr_\Sigma}\ln\left(\frac{1}{r_\Sigma}\int_0^{r_\Sigma}drr^2h(r)/h(r_\Sigma)\right). \tag{84}$$

Before analyzing specific models, some interesting conclusions can be obtained at this level of generality. One of these conclusions concerns the condition of bouncing at the surface which, of course, is related to the occurrence of a minimum radius A. According to Eq. (76) this requires $\Omega=0$, and we have

$$\frac{d^2A}{d\alpha^2}=F\frac{d\Omega}{d\alpha}, \tag{85}$$

or, using Eq. (81),

$$\frac{d\Omega}{d\alpha}(\Omega=0)=-\frac{F}{\tilde{\rho}_\Sigma}\left[R-\frac{2}{A}\left(P_{\perp\Sigma}-\frac{\hat{E}}{4\pi r_\Sigma^2}\right)\right]. \tag{86}$$

Observe that a positive energy flux ($\hat{E}$) tends to decrease the radius of the sphere, i.e., it favors the compactification of the object, which is easily understandable. The same happens when $R>0$ or $P_{\perp\Sigma}<0$. The opposite effect occurs when these quantities have the opposite signs. Now, for a positive energy flux the sphere can bounce at its surface only when

$$\frac{d\Omega}{d\alpha}(\Omega=0)\geqslant 0.$$

According to Eq. (86) this is equivalent to

$$-R(\Omega=0)+\frac{2P_{\perp\Sigma}}{A}\geqslant 0. \tag{87}$$

A physical meaning can be associated with this equation as follows. For a nonradiating, static configuration, R as defined by Eq. (82) consists of two parts, the first term, which together with $-[2(P_\perp-P_r)/r]_\Sigma$ represents the hydrodynamical force [see Eq. (43)], and the second, which is of course the gravitational force. The resulting force in the sense of increasing r is precisely $-R+[2(P_\perp-P_r)/r]_\Sigma$; if this is positive a net outward acceleration occurs, and vice

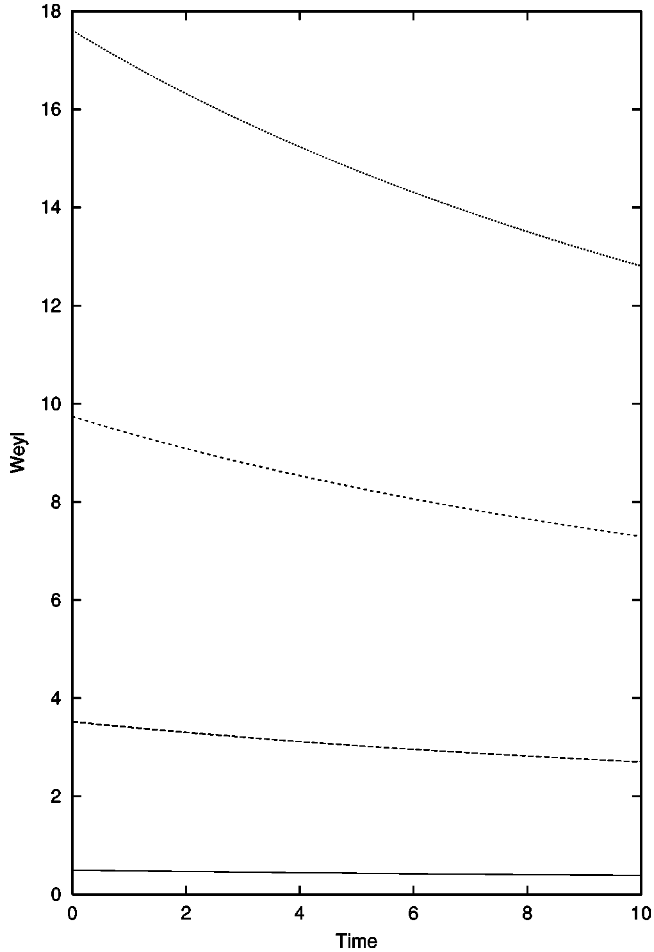

FIG. 5. Weyl tensor $W/m(0)$ as a function of time for the Schwarzschild-type model. The initial conditions are $A(0)=5$, $F(0)=0.6$, and $\Omega(0)=-0.1$. Curves represent different regions: $r/r_\Sigma=0.25$ (continuous line); 0.50 (dashed line); 0.75 (short-dashed line); and 1.00 (dotted line).

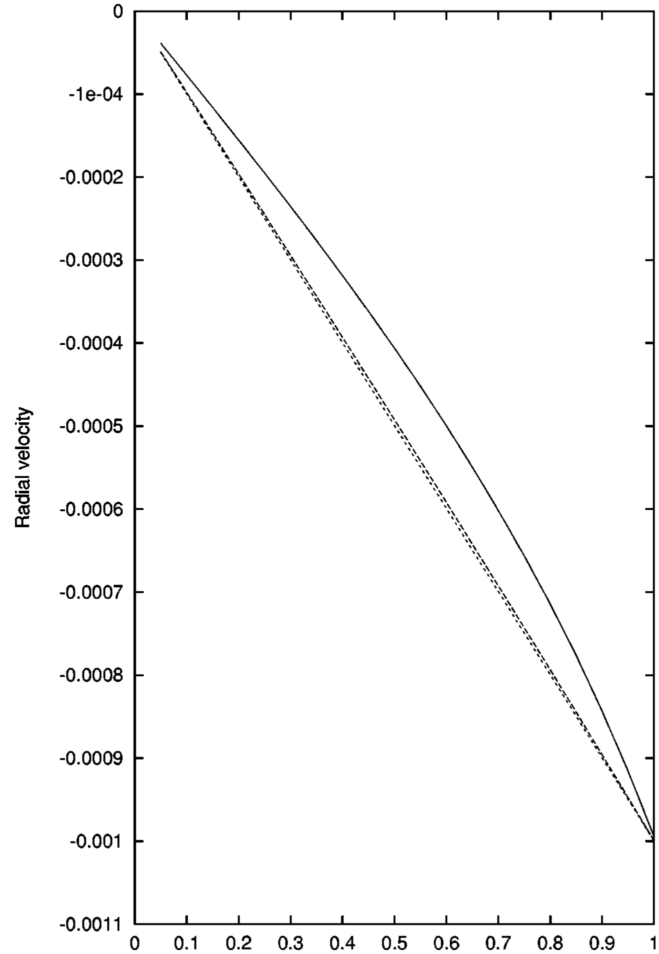

FIG. 6. Radial velocity ω as a function of r/r_Σ for $\alpha=10$. The initial velocity at the surface is -0.001. Curves represent different values of $F(0)$: 0.6 (continuous line); 0.96 (dashed line); and 0.996 (short-dashed line).

versa. Equation (87) is the natural generalization of this result for general nonstatic configurations.

As mentioned before, in addition to the surface equations, in some cases (depending on the type of matter under consideration) further information has to be provided in the form of an equation of state for the tangential stresses and/or a transport equation. In the next section we shall illustrate our method with three examples, one of which refers to an anisotropic fluid, and for which we shall further assume the equation of state (see [31,32])

$$P_\perp - P_r = \frac{C(\tilde{P}+\tilde{\rho})(4\pi r^3\tilde{P}+m)}{(r-2m)}, \tag{88}$$

where C is a constant.

IV. EXAMPLES

The only purpose of the present section is to illustrate the proposed method. For simplification we shall consider only the adiabatic case ($\epsilon=Q=0$). For all these models we shall calculate the physical and geometrical variables for any piece of matter, as functions of the timelike coordinate. In spite of the simplicity of the models, some interesting conclusions about the physical meaning of different variables may be reached.

One of the models has as the "seed" solution the well known Schwarzschild interior solution, whose properties have been extensively discussed in the literature. The second example is based on an anisotropic fluid without radial pressure. Models of this kind have also been discussed extensively since the original Einstein paper (see [33]). Finally, the third example represents the dynamic version of the Tolman type-VI static solution [34], whose equation of state, as is well known, approaches that for a highly compressed Fermi gas.

A. Schwarzschild-type model

This model is inspired by the well known interior Schwarzschild solution. Accordingly we take

$$\tilde{\rho}=f(t), \tag{89}$$

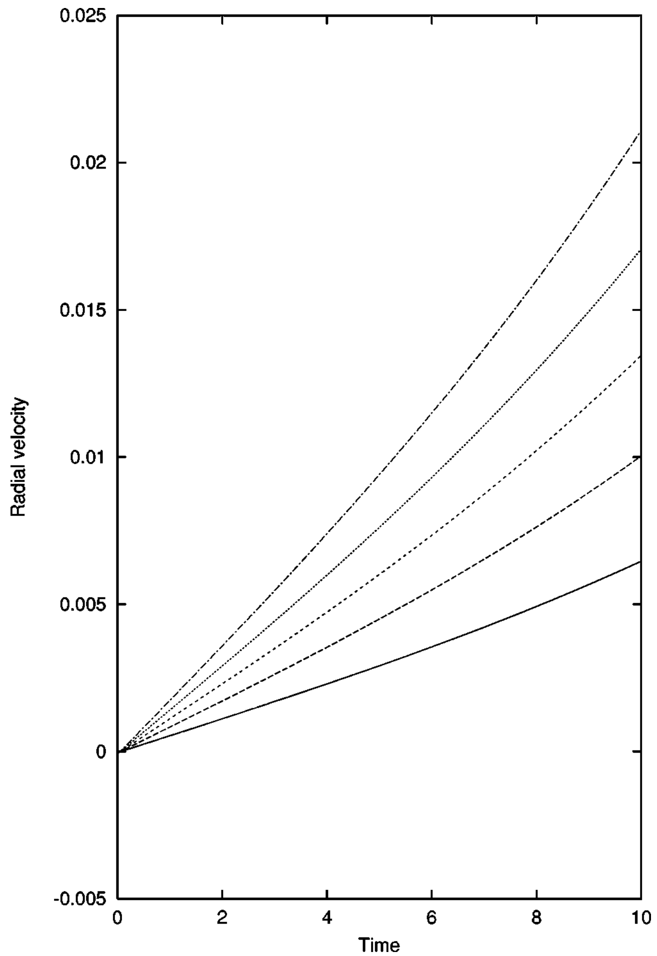

FIG. 7. Radial velocity ω as a function of time for the Tolman type-VI model. The initial conditions are $F(0)=0.581428528$ and $\Omega(0)=-0.0001$. Curves represent different regions: $r/r_\Sigma=0.2$ (continuous line); 0.4 (dashed line); 0.6 (short-dashed line); 0.8 (dotted line); and 1.0 (dot-dashed line).

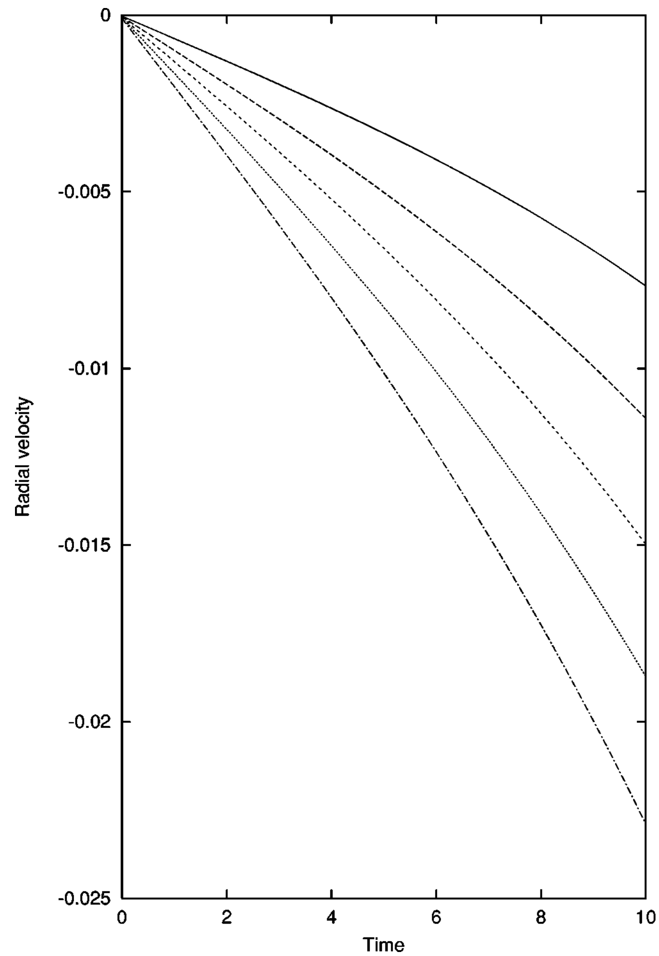

FIG. 8. Radial velocity ω as a function of time for the Tolman-type model. The initial conditions are $F(0)=0.561428547$ and $\Omega(0)=-0.0001$. Curves represent different regions: $r/r_\Sigma=0.2$ (continuous line); 0.4 (dashed line); 0.6 (short-dashed line); 0.8 (dotted line); and 1.0 (dot-dashed line).

where f is an arbitrary function of t. The expression for $\tilde{P}$ is

$$\frac{\tilde{P}+\frac{1}{3}\tilde{\rho}}{\tilde{P}+\tilde{\rho}}=\left(1-\frac{8\pi}{3}\tilde{\rho}r^2\right)^{1/2}k(t), \tag{90}$$

where k is a function of t to be determined from the junction conditions (40), which in terms of effective variables becomes

$$\tilde{P}_\Sigma=\tilde{\rho}_\Sigma\Omega^2. \tag{91}$$

Thus, using Eqs. (90) and (91) we have for the effective variables

$$\tilde{\rho}=\frac{3(1-F)}{8\pi r_\Sigma^2}, \tag{92}$$

$$\tilde{P}=\frac{\tilde{\rho}}{3}\left\{\frac{\chi F^{1/2}-3\psi\xi}{\psi\xi-\chi F^{1/2}}\right\}, \tag{93}$$

with

$$\xi=[1-(1-F)(r/r_\Sigma)^2]^{1/2},$$

and

$$\chi=3(\Omega^2+1)(1-F),$$

$$\psi=(3\Omega^2+1)(1-F).$$

For the metric functions m and ν we get, using Eqs. (74) and (75)

$$m=m_\Sigma(r/r_\Sigma)^3, \tag{94}$$

$$e^\nu=\left\{\frac{\chi F^{1/2}-\psi\xi}{2(1-F)}\right\}^2. \tag{95}$$

The third surface equation for this model becomes

$$\frac{d\Omega}{d\alpha}=\frac{\Omega^2}{2A}[7-3\Omega^2+3F(\Omega^2-1)]. \tag{96}$$

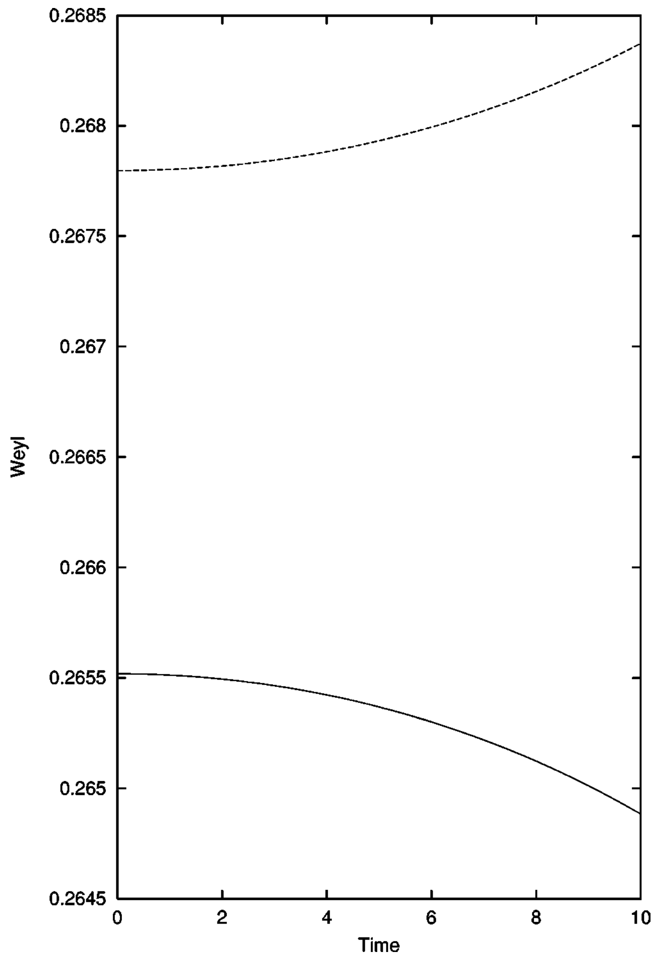

FIG. 9. Weyl tensor $W/m(0)$ at $r/r_{\Sigma}=0.4$ as a function of time for the Tolman type-VI type model. The initial conditions are $F(0)=0.581428528$ (dashed line); $F(0)=0.561428547$ (continuous line); and $\Omega(0)=-0.0001$.

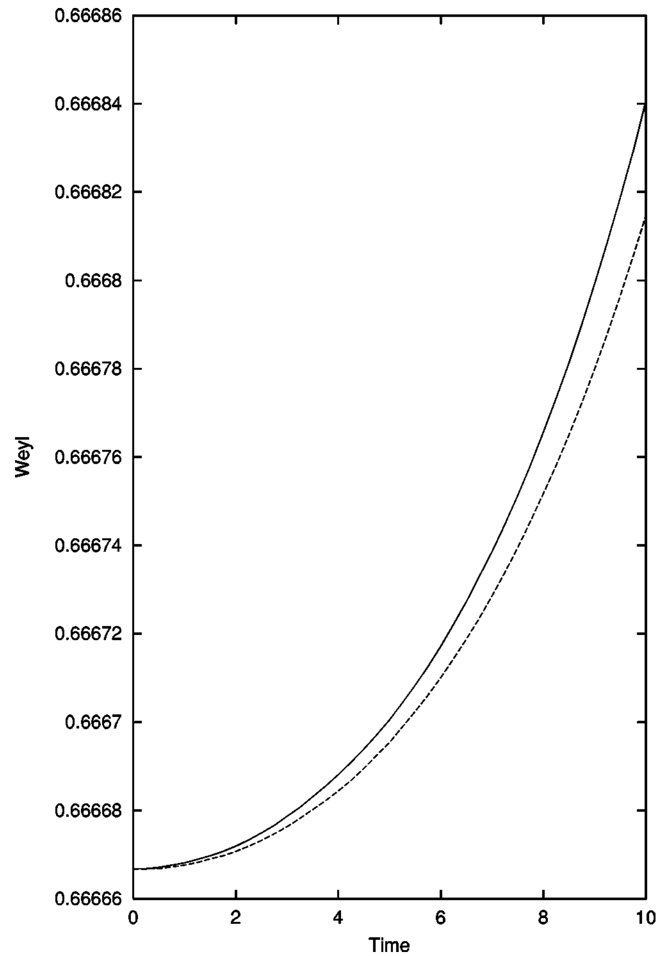

FIG. 10. Weyl tensor $W/m(0)$ at $r/r_{\Sigma}=1.0$ as a function of time for the Tolman type-VI type model. The initial conditions are $F(0)=0.581428528$ (dashed line); $F(0)=0.561428547$ (continuous line); and $\Omega(0)=-0.0001$.

This equation together with Eqs. (76) and (79) form the set of surface equations for this model. We have integrated it numerically and from this integration all physical variables are found for any piece of the fluid distribution, following the algorithm described above.

Figures 1–5 exhibit the behavior of ρ, P, ω, σ, and W for an initially contracting configuration, as functions of α and different pieces of matter. Figure 6 shows the profile of ω as a function of r/r_{Σ} for $\alpha=10$.

B. Lemaitre-Florides-type model

This model has as the "seed" solution a configuration with homogeneous energy density and vanishing radial pressure. Configurations of this kind were suggested for the first time by Lemaitre [35].

The corresponding effective variables now are

$$\tilde{\rho}=f(t) \tag{97}$$

and

$$\tilde{P}=0. \tag{98}$$

Observe that in this case, because of Eqs. (97) and (98), it follows from Eqs. (72) and (73) that the radial pressure is discontinuous at the boundary surface, with

$$P_{r\Sigma}=-3(1-F)\Omega^2/8\pi r_{\Sigma}^2, \tag{99}$$

for otherwise either ρ_{Σ} or Ω should vanish at Σ. Therefore the only way to "dynamize" this model is by relaxing boundary conditions, allowing for the presence of a kind of surface tension.

Once the effective variables are defined, we need only the value of the tangential pressure at the boundary to close the system of surface equations. This is obtained by evaluating Eq. (88) at Σ.

Next, following the algorithm, all physical variables may be found for any piece of material as functions of the timelike coordinate. Although we are not going to exhibit them here, because the graphics are not particularly illuminating, we wanted to present an example that, in addition to the fact that it implies an anisotropic fluid, requires the introduction of a surface tension to allow the application of the algorithm.

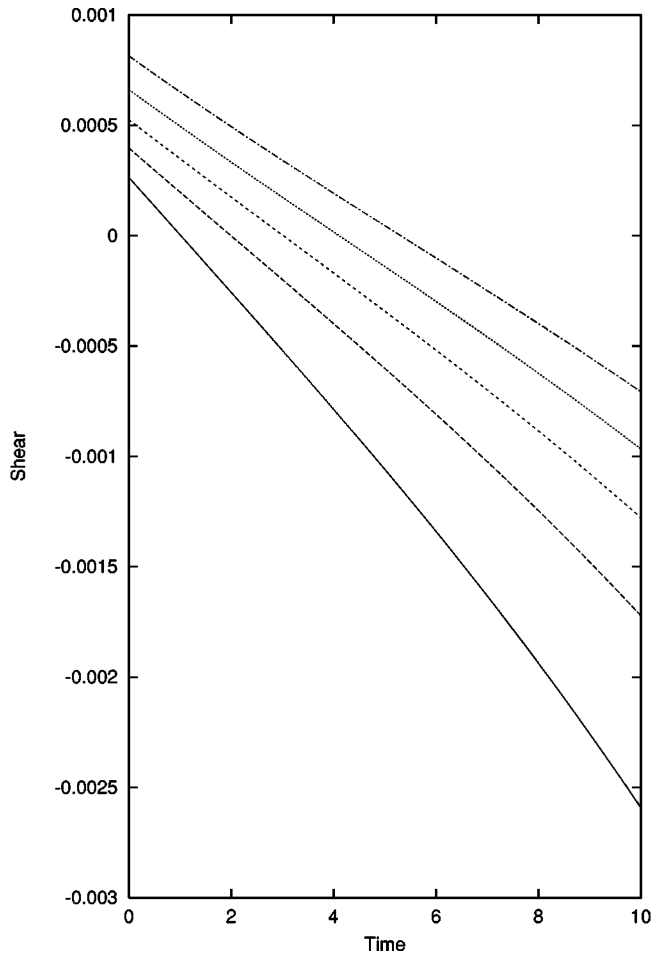

FIG. 11. Shear $\sigma m(0)$ as a function of time for the Tolman type-VI type model. The initial conditions are $F(0)=0.581428528$ and $\Omega(0)=-0.0001$. Curves represent different regions: $r/r_\Sigma=0.2$ (continuous line); 0.4 (dashed line); 0.6 (short-dashed line); 0.8 (dotted line); and 1.0 (dot-dashed line).

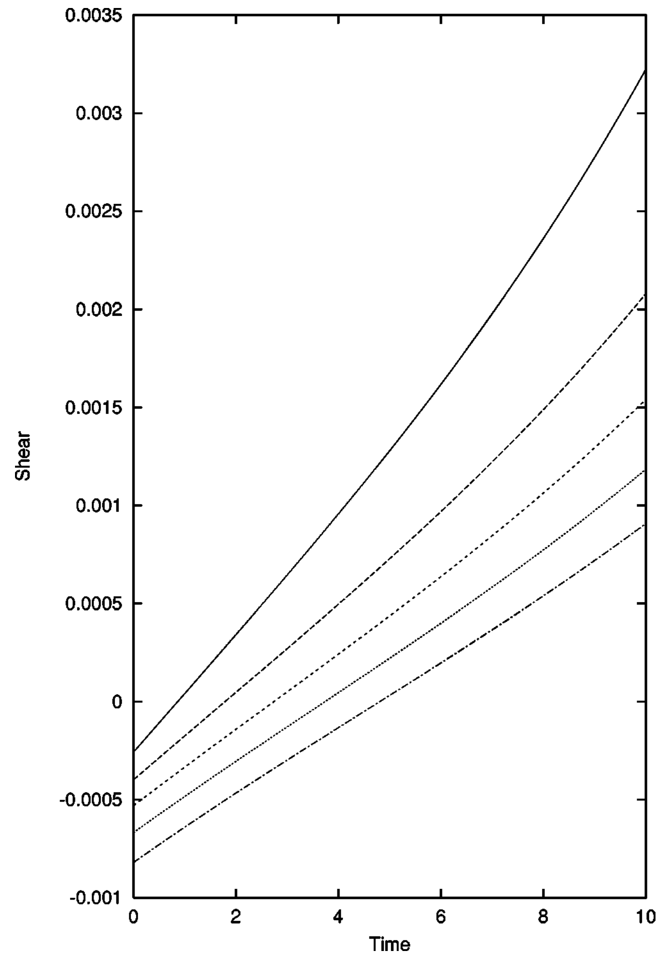

FIG. 12. Shear $\sigma m(0)$ as a function of time for the Tolman-type model. The initial conditions are $F(0)=0.561428547$ and $\Omega(0)=-0.0001$. Curves represent different regions: $r/r_\Sigma=0.2$ (continuous line); 0.4 (dashed line); 0.6 (short-dashed line); 0.8 (dotted line); and 1.0 (dot-dashed line).

C. Tolman type-VI model

Our last example is based on the Tolman type-VI solution. Accordingly the effective variables for this model will be

$$\tilde{\rho}=\frac{3g(t)}{r^2} \tag{100}$$

and

$$\tilde{P}=\frac{g[9-bK(r/r_\Sigma)]}{[9-b(r/r_\Sigma)]r^2}, \tag{101}$$

where g and b are functions of α, to be obtained from Eq. (91). Then,

$$\tilde{\rho}=\frac{3(1-F)}{24\pi r^2}. \tag{102}$$

Using Eqs. (74) and (75) we get

$$m=m_\Sigma r/r_\Sigma, \tag{103}$$

$$\nu=\ln F+\frac{8\pi g}{F}\left\{4\ln(r/r_\Sigma)+8\ln\left(\frac{b(r/r_\Sigma)-K}{b-9}\right)\right\}. \tag{104}$$

Finally, solving the surface equations for this model, m and ν are completely determined and all physical variables can thereby be calculated. In addition to the intrinsic physical interest of the equation of state of this "seed" model mentioned before, it is interesting because of the fact that the static limit of the model (unlike the previous ones) is "unstable," in the sense that it requires a specific value of the gravitational potential at the boundary, namely, $m_\Sigma(0)/r_\Sigma=3/14$. For values above (below) this, the sphere starts to collapse (expand).

Figures 7 and 8 display the evolution of velocity (ω) for different regions of the sphere, and for initial values of F corresponding to values of $m_\Sigma(0)/r_\Sigma$ above and below the equilibrium value, respectively. Figures 9 and 10, represent the evolution of the Weyl tensor (W) for some internal region and the boundary surface, respectively, and initial values of F corresponding to values of $m_\Sigma(0)/r_\Sigma$ above and below equilibrium. Finally, Figs. 11 and 12 exhibit the behavior of the shear (σ) for different regions and initial values of F corresponding to values of $m_\Sigma(0)/r_\Sigma$ above and below equilibrium.

We shall comment on these graphics in the next section.

V. CONCLUSIONS

A method has been presented that allows for the description of radiating self-gravitating relativistic spheres. In its most general form, the approach incorporates the two limiting cases of radiation transport (free streaming and diffusion) as well as the possibility of dealing with anisotropic fluids.

The cornerstone of the algorithm is an ansatz based on a specific definition of the post-quasistatic approximation, namely: considering different degrees of departure from equilibrium, the post-quasistatic regime (i.e., the next step after the quasistatic situation) is defined as that characterized by metric functions whose radial dependence is the same as that of the quasistatic regime. This in turn implies, that the effective variables defined above share the same radial dependence as the correspondig physical variables of the quasistatic regime. The rationale behind this definition seems intelligible when it is remembered that in the latter case (the quasistatic) the effective variables share the same radial dependence as that of the physical variables in the static regime. Thus, starting with a static configuration, the first ''level'' of equilibrium, beyond the quasistatic situation, is represented by the post-quasistatic regime.

Once the static (''seed'') solution has been selected, then the definition of the effective variables together with surface equations allows for determination of the metric functions, which in turn lead to the full description of the physical variables as functions of the timelike coordinate for any region of the sphere. In this process, depending on the kind of matter and/or the prevailing transport approximation, additional equations of state and/or transport equations and/or some of the surface variables (e.g., the luminosity) have to be specified.

Once all physical variables have been found (particularly the energy density and the radial pressure) then we may, in principle, go to the next step, assuming that the effective variables now share the same radial dependence as that of the physical variables just obtained. In this sense the algorithm may be regarded as an iterative approach. For obvious reasons we have restricted ourselves to the first step of the process. It remains to be seen if available physical evidence justifies going through the complexities associated with the ''post-post-quasistatic'' approximation.

In order to illustrate the method, and without the pretension of modeling specific astrophysical scenarios, we have presented three examples, in the simplest (adiabatic) case.

In the first model, the profiles of the shear and the Weyl tensor clearly illustrate the ''dynamics'' of the model, tending to zero in the static limit. The fact that these two quantities vanish in the quasistatic regime (for this specific model) further brings out their relevance in the treatment of situations off equilibrium. On the other hand however, the velocity profiles show almost no difference between the two regimes. Deviations from homology contraction due to relativistic gravitational effects are also indicated.

The purpose of the second example was to illustrate the implementation of the algorithm for anisotropic fluids. The very particular form of the ''seed'' equation of state of this model imposes discontinuity (surface tension) of the radial pressure at the boundary. Of course this discontinuity vanishes in the static (or quasistatic) regime.

Finally, a model based on the Tolman type-VI solution was presented. This static solution, as was already mentioned, requires a specific value of $m_\Sigma(0)/r_\Sigma$; accordingly, any deviation from this value leads to deviations from the static regime (observe that the quasistatic regime is incompatible with this solution). The velocity profiles indicate that all regions either expand or contract, and therefore cracking (different signs of the velocity for different regions of the sphere) will not occur [36]. This is consistent with the established fact that cracking occurs only for anisotropic fluids or isotropic fluids with outgoing radiation in the free streaming approximation.

Also, the profiles of the Weyl tensor and the shear, clearly diverging from the initial values as time proceeds and the evolution becomes more and more ''dynamic,'' stress once again their roles in describing departures from equilibrium.

ACKNOWLEDGMENTS

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[1] J. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).
[2] L. Lenher, Class. Quantum Grav. **18**, R25 (2001).
[3] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[4] J. A. Font, Living Rev. Relativ. **47**, 2 (2000).
[5] J. Winicour, Living Rev. Relativ. **1**, 5 (1998).
[6] W. Bonnor, A. Oliveira, and N. O. Santos, Phys. Rep. **181**, 269 (1989); M. Govender, S. Maharaj, and R. Maartens, Class. Quantum Grav. **15**, 323 (1998); D. Schafer and H. Goenner, Gen. Relativ. Gravit. **32**, 2119 (2000); M. Govender, R. Maartens, and S. Maharaj, Phys. Lett. A **283**, 71 (2001); S. Wagh *et al.*, Class. Quantum Grav. **18**, 2147 (2001).
[7] W. Barreto, H. Martínez, and B. Rodríguez (unpublished).
[8] L. Herrera, J. Jimenez, and G. Ruggeri, Phys. Rev. D **22**, 2305 (1980); L. Herrera and L. Nunez, Fundam. Cosmic **14**, 235 (1990).
[9] D. Kazanas and D. Schramm, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).

[10] W. D. Arnett, Astrophys. J. **218**, 815 (1977).
[11] D. Kazanas, Astrophys. J. **222**, 2109 (1978).
[12] J. Lattimer, Nucl. Phys. **A478**, 199 (1988).
[13] M. May and R. White, Phys. Rev. **141**, 1232 (1966); J. Wilson, Astrophys. J. **163**, 209 (1971); A. Burrows and J. Lattimer, *ibid.* **307**, 178 (1986); R. Adams, B. Cary, and J. Cohen, Astrophys. Space Sci. **155**, 271 (1989).
[14] W. Bonnor and H. Knutsen, Int. J. Theor. Phys. **32**, 1061 (1993).
[15] H. Bondi, Proc. R. Soc. London **A281**, 39 (1964).
[16] N. O. Santos, Mon. Not. R. Astron. Soc. **216**, 403 (1985).
[17] R. Bowers and E. Liang, Astrophys. J. **188**, 657 (1974).
[18] R. Tolman, Phys. Rev. **35**, 875 (1930).
[19] F. I. Cooperstock, R. S. Sarracino, and S. S. Bayin, J. Phys. A **14**, 181 (1981).
[20] J. Devitt and P. S. Florides, Gen. Relativ. Gravit. **21**, 585 (1989).
[21] L. Herrera, A. Di Prisco, J. L. Hernandez-Pastora, and N. O. Santos, Phys. Lett. A **237**, 113 (1998).
[22] L. Herrera and N. O. Santos, Gen. Relativ. Gravit. **27**, 1071 (1995).
[23] Ø. Grøn, Phys. Rev. D **31**, 2129 (1985).
[24] A. Lightman, W. Press, R. Price, and S. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton, NJ, 1975).
[25] R. Penrose, in *General Relativity, An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), pp. 581–638.
[26] J. Wainwright, Gen. Relativ. Gravit. **16**, 657 (1984); S. W. Goode and J. Wainwright, Class. Quantum Grav. **2**, 99 (1985); W. B. Bonnor, Phys. Lett. **112A**, 26 (1985); **122**, 305 (1987); S. W. Goode, A. Coley, and J. Wainwright, Class. Quantum Grav. **9**, 445 (1992); N. Pelavas and K. Lake, Phys. Rev. D **62**, 044009 (2000); L. Herrera *et al.*, J. Math. Phys. **42**, 2199 (2001).
[27] F. Mena and R. Tavakol, Class. Quantum Grav. **16**, 435 (1999); D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979); D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984); R. P. A. C. Newman, Class. Quantum Grav. **3**, 527 (1986); B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988); I. Dwivedi and P. Joshi, Class. Quantum Grav. **9**, L69 (1992); P. Joshi and I. Dwivedi, Phys. Rev. D **47**, 5357 (1993); T. P. Singh and P. Joshi, Class. Quantum Grav. **13**, 559 (1996); L. Herrera and N. O. Santos, Phys. Rep. **286**, 53 (1997); H. Bondi, Mon. Not. R. Astron. Soc. **262**, 1088 (1993); W. Barreto, Astrophys. Space Sci. **201**, 191 (1993); A. Coley and B. Tupper, Class. Quantum Grav. **11**, 2553 (1994); J. Martinez, D. Pavon, and L. Nunez, Mon. Not. R. Astron. Soc. **271**, 463 (1994); T. Singh, P. Singh, and A. Helmi, Nuovo Cimento Soc. Ital. Fis., B **110**, 387 (1995); A. Das, N. Tariq, and J. Biech, J. Math. Phys. **36**, 340 (1995); R. Maartens, S. Maharaj, and B. Tupper, Class. Quantum Grav. **12**, 2577 (1995); A. Das, N. Tariq, D. Aruliah, and T. Biech, J. Math. Phys. **38**, 4202 (1997); E. Corchero, Class. Quantum Grav. **15**, 3645 (1998); E. Corchero, Astrophys. Space Sci. **259**, 31 (1998); H. Bondi, Mon. Not. R. Astron. Soc. **302**, 337 (1999); H. Hernandez, L. Nunez, and U. Percoco, Class. Quantum Grav. **16**, 897 (1999); T. Harko and M. Mark, J. Math. Phys. **41**, 4752 (2000); A. Das and S. Kloster, Phys. Rev. D **62**, 104002 (2000); P. Joshi, N. Dadhich, and R. Maartens, gr-qc/0109051.
[28] M. Schwarzschild, *Structure and Evolution of the Stars* (Dover, New York, 1958); R. Kippenhahn and A. Weigert, *Stellar Structure and Evolution* (Springer-Verlag, Berlin, 1990); C. Hansen and S. Kawaler, *Stellar Interiors: Physical Principles, Structure and Evolution* (Springer-Verlag, Berlin, 1994).
[29] I. Iben, Astrophys. J. **138**, 1090 (1963).
[30] E. Myra and A. Burrows, Astrophys. J. **364**, 222 (1990).
[31] W. Barreto, Astrophys. Space Sci. **201**, 191 (1993).
[32] M. Cosenza, L. Herrera, M. Esculpi, and L. Witten, Phys. Rev. D **22**, 2527 (1982).
[33] A. Einstein, Ann. Math. **40**, 4 (1939); P. Florides, Proc. R. Soc. London **A337**, 529 (1974); B. K. Datta, Gen. Relativ. Gravit. **1**, 19 (1970); H. Bondi, *ibid.* **2**, 321 (1971); A. Evans, *ibid.* **8**, 155 (1977).
[34] R. Tolman, Phys. Rev. **55**, 364 (1939).
[35] G. Lemaitre, Ann. Bull. Soc. R. Sci. Med. Nat. Bruxelles **A53**, 51 (1933).
[36] L. Herrera, Phys. Lett. A **165**, 206 (1992).