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A source of a quasi-spherical space–time: The case for the *M–Q* solution

Abstract We present a physically reasonable source for an static, axially-symmetric solution to the Einstein equations. Arguments are provided, supporting our belief that the exterior space-time produced by such source, describing a quadrupole correction to the Schwarzschild metric, is particularly suitable (among known solutions of the Weyl family) for discussing the properties of quasi-spherical gravitational fields.

Keywords Axially-symmetric solutions · Weyl family · Source

1 Introduction

As is well known, Weyl solutions [1] represent the family of all static and axially-symmetric exterior solutions to the Einstein equations. Since there are as many different Weyl solutions as there are different harmonic functions (see next section), then the obvious question arises: what is the exact vacuum solution to the Einstein equations corresponding to a given non-spherical, static axially symmetric source (an ellipsoid, say)?

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If the field is not particularly intense ($r \gg 2M$) and the deviation from spherical symmetry is slight, then there is not problem in representing the corresponding field (both inside and outside the source) as a suitable perturbation of the spherically symmetric exact solution. However, as the object becomes more and more compact, such perturbative scheme will eventually fail.

Indeed, as it is well known [2], the only static and asymptotically-flat vacuum space-time possessing a regular horizon is the Schwarzschild solution. For all the others Weyl exterior solutions [1], the physical components of the Riemann tensor exhibit singularities at $r = 2M$. Therefore, it is intuitively clear that as r approaches $2M$ and gravitational field becomes stronger, the properties of sources of Weyl space-time should start drastically to differ from the properties of spherical sources [3]. It is important to keep in mind that this sharp difference in the behaviour of both types of sources (for very high gravitational fields) is independent on how small, multipole moments (higher than monopole) of the Weyl source, are. This is so because, as r approaches $2M$, any finite perturbation of the Schwarzschild space-time becomes fundamentally different from any Weyl solution, even when the latter is characterized by parameters whose values are arbitrarily close to those corresponding to the spherical symmetry. This point has been stressed before [4], but usually it has been overlooked.

Furthermore in a recent work [5] it was shown that for a non-spherical source (even in the case of slight deviations from spherical symmetry), the speed of entering the collapse regime decreases substantially, as compared with the exactly spherically symmetric case. In the same order of ideas it has been shown [6] that small departures from sphericity, produce significant decreasing (increasing) in the values of active gravitational mass of collapsing (expanding) spheres, with respect to its value in equilibrium, enhancing thereby the stability of the system. Also, the sensitivity of the trajectories of test particles in the γ spacetime, to small changes of γ, for orbits close to $2M$, has been brought out [7]. It is important to stress that all these effects take place for strong gravitational fields, but for $r > 2M$.

These works [5–7] were done using as Weyl solution, the so-called gamma metric (γ-metric) [8]. This metric, which is also known as Zipoy–Vorhees metric, belongs to the family of Weyl's solutions, and is continuously linked to the Schwarzschild space-time through one of its parameters. The motivation for this choice was that the exterior γ-metric corresponds to a solution of the Laplace equation (in cylindrical coordinates) with the same singularity structure as the Schwarzschild solution (a line segment [8]). In this sense the γ-metric appears as the "natural" generalization of Schwarzschild space-time to the axisymmetric case.

However, the existence of so many different (physically distinguishable [9]) Weyl solutions gives rise to the question: which among Weyl solutions is better entitled to describe small deviations from spherical symmetry?

Although it should be obvious that such a question does not have a unique answer (there is an infinite number of ways of being non-spherical, so to speak), we shall invoke a very simple criterion, emerging from Newtonian gravity, in order to choose our solution.

Indeed, in order to answer the question above, let us recall [10] that most known Weyl solutions, present a drawback when describing quasi-spherical

space-times. It consists of the fact that its multipole structure is such that multipole moments, higher than quadrupole, are of the same order as the quadrupole moment. Instead, as it is intuitively clear, the relevance of such multipole moments should decrease as we move from lower to higher moments, the quadrupole moment being the most relevant for a small departure from sphericity. Thus for example in Newtonian gravity, multipole moments of an ellipsoid of rotation, with homogeneous density, and axes (a,a,b) read:

$$\begin{cases} D_{2n} = (-2)^n 3 M a^{2n} \epsilon^n (1 - \epsilon/2)^n / [(2n+1)(2n+3)], \quad \epsilon \equiv (a-b)/a, \\ D_{2n+1} = 0 \end{cases},$$

because of the factor ϵ^n, this equation clearly exhibits the progressive decreasing of the relevance of multipole moments as n increases.

Thus, in order to describe small departures from sphericity, by means of exact solution to the Einstein equations, we would require an exact solution whose multipole structure shares the property mentioned above. Fortunately enough, such solution exists [10]. Indeed, there is one (exact) solution of the Weyl family, which may be interpreted as a quadrupole correction to the Schwarzschild space-time (see below).

It is for this exterior metric that we are going to construct a source. The motivation for this is twofold: On the one hand, it is always interesting to propose bounded and physically reasonable sources of gravitational fields, which may serve as models of compact object. On the other hand, spherical symmetry is a common assumption in the study of compact self-gravitating objects (white dwarfs, neutron stars, black holes). Therefore it is pertinent to ask, how do small deviations from this assumption, related to any kind of perturbation (e.g. fluctuations of the stellar matter, external perturbations, etc.), affect the properties of the system? However, as mentioned before, for sufficiently strong fields, in order to answer to this question it is necessary to deal with non-spherically symmetric exact solution to the Einstein equations.

For constructing the source we shall follow a prescription given by Hernández [11] allowing to obtain interior solutions of Weyl space-time, from known spherically symmetric interior solutions. Our interior solution will be obtained from the interior Schwarzschild solution (homogeneous density).

The paper is organized as follows. In the next section we review Weyl solutions and the concept of relativistic multipoles. In Sect. 3 we describe the M–Q solution [10] and give its properties. The Hernández method is applied in Sect. 4 to obtain an interior to the M–Q solution. Finally, results are discussed in the last section.

2 The Weyl metrics

Static axisymmetric solutions to the Einstein equations are given by the Weyl metric [1]

$$ds^2 = -e^{2\Psi} dt^2 + e^{-2\Psi} [e^{2\Gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{1}$$

where metric functions have to satisfy

$$\Psi_{,\rho\rho} + \rho^{-1} \Psi_{,\rho} + \Psi_{,zz} = 0 \tag{2}$$

and

$$\Gamma_{,\rho} = \rho\left(\Psi_{,\rho}^2 - \Psi_{,z}^2\right); \quad \Gamma_{,z} = 2\rho\Psi_{,\rho}\Psi_{,z}. \tag{3}$$

Observe that (2) is just the Laplace equation for Ψ (in the Euclidean space), and furthermore it represents the integrability condition for (3), implying that for any "Newtonian" potential we have a specific Weyl metric, a well known result.

The general solution of the Laplace Eq. (2) for the function Ψ, presenting an asymptotically flat behaviour, results to be

$$\Psi = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos\theta), \tag{4}$$

where $r = (\rho^2 + z^2)^{1/2}$, $\cos\theta = z/r$ are Weyl spherical coordinates and $P_n(\cos\theta)$ are Legendre Polynomyals. The coefficients a_n are arbitrary real constants which have been named in the literature "Weyl moments", although they cannot be identified as relativistic multipole moments in spite of the formal similarity between expression (4) and the Newtonian potential.

Another interesting way of writing the solution (4) was obtained by Erez–Rosen [12] and Quevedo [13], integrating Eqs. (2) and (3) in prolate spheroidal coordinates, which are defined as follows

$$\begin{aligned} x &= \frac{r_+ + r_-}{2\sigma}, \quad y = \frac{r_+ - r_-}{2\sigma}, \\ r_\pm &\equiv [\rho^2 + (z \pm \sigma)^2]^{1/2}, x \geq 1, \quad -1 \leq y \leq 1, \end{aligned} \tag{5}$$

where σ is an arbitrary constant which will be identified later with the Schwarzschild's mass. Inverse relation between both families of coordinates is given by

$$\begin{aligned} \rho^2 &= \sigma^2(x^2 - 1)(1 - y^2), \\ z &= \sigma\, x\, y. \end{aligned} \tag{6}$$

The prolate coordinate x represents a radial coordinate, whereas the other coordinate, y represents the cosine function of the polar angle.

In these prolate spheroidal coordinates Ψ takes the form

$$\Psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n Q_n(x) P_n(y), \tag{7}$$

being $Q_n(y)$ Legendre functions of second kind and q_n a set of arbitrary constants. The corresponding expression for the function Γ, has been obtained by Quevedo [13].

A sub-family of Weyl solutions has been obtained by Gutsunaev and Manko [14–16] starting from the Schwarzschild solution as a "seed" solution.

Both sets of coefficients, a_n and q_n, characterize any Weyl metric [13]. Nevertheless, as mentioned before, these constants do not give us physical information about the metric since they do not represent the "real" multipole moments of the source. That is not the case for the relativistic multipole moments firstly defined by Geroch [17], and subsequently by Hansen [18] and Thorne [19], which, as it

is known, characterize completely and uniquely, at least in the neighbourhood of infinity, every asymptotically flat and stationary vacuum solution [20], providing at the same time a physical description of the corresponding solution.

An algorithm to calculate the Geroch multipole moments was developed by Fodor, Hoenselaers and Perjes [21] (FHP). By applying such method, the resulting multipole moments of the solution are expressed in terms of the Weyl moments. Similar results are obtained from the Thorne's definition, using harmonic coordinates. The structure of the obtained relation between coefficients a_n and these relativistic moments allows to express the Weyl moments as a combination of the Geroch relativistic moments.

2.1 The Monopole–Quadrupole solution, M–Q

In this section we would like to describe the properties of a solution (hereafter referred to as the M–Q solution [10]) which is particularly suitable for the study of perturbations of the spherical symmetry. The main argument to support this statement is based on the fact that previously known Weyl metrics (e.g. Gutsunaev–Manko [14], Manko [15], gamma metric [8], Curzon [22], etc.) have a multipolar structure (in the Geroch sense) such that all the moments higher than the quadrupole, are of the same order as the quadrupole. In fact for the above mentioned metrics we have (odd moments are of course vanishing)

$$
\begin{aligned}
M_0^{\mathrm{GM}} &= M_0^{\mathrm{ER}} = M, \\
M_2^{\mathrm{GM}} &= M_2^{\mathrm{ER}} = \frac{2}{15} q_2 M^3, \\
M_4^{\mathrm{GM}} &= -3 M_4^{\mathrm{ER}} = \frac{4}{35} q_2 M^5, \\
M_6^{\mathrm{GM}} &= M_6^{\mathrm{ER}} - \frac{2}{7} \frac{817}{33} M^2 M_4^{\mathrm{ER}} = \frac{2}{15} \frac{4}{231} q_2 M^7 \left(\frac{194}{7} + \frac{14}{15} q_2 \right),
\end{aligned}
$$

where q_2 is the quadrupole parameter in the Erez-Rosen metric. For the gamma metric results to be

$$
\begin{aligned}
M_0 &= \gamma M, \\
M_2 &= \gamma \frac{M^3}{3} (1 - \gamma^2), \\
M_4 &= \gamma \frac{M^5}{5} (1 - \gamma^2) \left(1 - \frac{19}{21} \gamma^2 \right), \\
M_6 &= \gamma \frac{M^7}{7} (1 - \gamma^2) \left(1 + \frac{389}{495} \gamma^4 - \frac{292}{165} \gamma^2 \right),
\end{aligned}
$$

and finally, Curzon metric is the worst case of the mentioned metrics since it possesses a unique parameter which represents the mass, and all higher moments

are proportional to increasing powers of that parameter, i.e.,

$$M_0 = -a_0,$$
$$M_2 = \frac{1}{3}a_0^3,$$
$$M_4 = -\frac{19}{105}a_0^5,$$
$$M_6 = \frac{389}{3465}a_0^7.$$

In [10] it was shown that it is possible to find a solution of the Weyl family, by a convenient choice of coefficients a_n, such that the resulting solution possesses only monopole and quadrupole moments (in the Geroch sense) [17]. The obtained solution (M–Q) may be written as a finite series of Gutsunayev-Manko [14] and Erez-Rosen [12] solutions, as follows:

$$\Psi_{M-Q} = \Psi_{q^0} + q\Psi_{q^1} + q^2\Psi_{q^2} + \cdots = \sum_{\alpha=0}^{\infty} q^\alpha \Psi_{q^\alpha}, \qquad (8)$$

where the zeroth order corresponds to the Schwarzschild solution.

$$\Psi_{q^0} = -\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+1} P_{2n}(\cos\theta), \qquad (9)$$

with $\lambda \equiv M/r$ and it appears that each power in q adds a quadrupole correction to the spherically symmetric solution. Now, it should be observed that due to the linearity of Laplace equation, these corrections give rise to a series of exact solutions. In other words, the power series of q may be cut at any order, and the partial summatory, up to that order, gives an exact solution representing a quadrupolar correction to the Schwarzschild solution.

The simplest way to interpret physically the exact solutions obtained from the quadrupolar corrections described above, consists in analyzing the corresponding multipolar structure. Thus , it can be shown that cutting solution (8) at some order α, one obtains an exact solution with the following properties:

- Both, the monopole and the quadrupole moments are non-vanishing: $M_0 \equiv M$, $M_2 \equiv qM^3$.
- All the remaining moments until order $2(\alpha + 1)$ (included) vanish.
- All moments above the $2(\alpha+1)$-pole are of order $q^{\alpha+1}$. Therefore, the solution represents a quadrupolar correction to the Schwarzschild solution, which is an exact solution up to the given order.

To illustrate further our point, let us present the explicit solution up to first order, describing a quadrupolar correction to the monopole. The corresponding metric

functions read (note a misprint in the Eq. (13) in [23])

$$\Psi_{M-Q}^{(1)} \equiv \Psi_{q^0} + q\Psi_{q^1} = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{5}{8}q(3y^2-1)$$
$$\times \left[\left(\frac{3x^2-1}{4} - \frac{1}{3y^2-1}\right)\ln\left(\frac{x-1}{x+1}\right)\right.$$
$$\left. - \frac{2x}{(x^2-y^2)(3y^2-1)} + \frac{3x}{2}\right], \tag{10}$$

$$\Gamma_{M-Q}^{(1)} \equiv \Gamma_{q^0} + q\Gamma_{q^1} + q^2\Gamma_{q^2} = \frac{1}{2}\left(1 + \frac{225}{24}q^2\right)\ln\left(\frac{x^2-1}{x^2-y^2}\right)$$
$$+ \frac{225}{1024}q^2(x^2-1)(1-y^2)(x^2+y^2-9x^2y^2-1)\ln^2\left(\frac{x-1}{x+1}\right)$$
$$- \frac{15}{4}q(1-y^2)\left[1 - \frac{15}{64}q(x^2+4y^2-9x^2y^2+4)\right]$$
$$- \frac{75}{16}q^2x^2\frac{1-y^2}{x^2-y^2} - \frac{5}{4}q(x^2+y^2)\frac{1-y^2}{(x^2-y^2)^2}$$
$$- \frac{75}{192}q^2(2x^6 - x^4 + 3x^4y^2 - 6x^2y^2 + 4x^2y^4 - y^4 - y^6)\frac{1-y^2}{(x^2-y^2)^4}$$
$$- \frac{15}{8}qx(1-y^2)\left[1 - \frac{15}{32}q\left(x^2+7y^2-9x^2y^2+1-\frac{8}{3}\frac{x^2+1}{x^2-y^2}\right)\right]$$
$$\times \ln\left(\frac{x-1}{x+1}\right). \tag{11}$$

The first 12 Geroch moments of this solution are (odd moments vanish because of the reflexion symmetry)

$$M_0 = M, \quad M_2 = M^3q, \quad M_4 = 0, \quad M_6 = -\frac{60}{77}M^7q^2,$$
$$M_8 = -\frac{1060}{3003}M^9q^2 - \frac{40}{143}M^9q^3, \quad M_{10} = -\frac{19880}{138567}M^{11}q^2 + \frac{146500}{323323}M^{11}q^3,$$
$$M_{12} = -\frac{23600}{437437}M^{13}q^2 + \frac{517600}{1062347}M^{13}q^3 + \frac{4259400}{7436429}M^{13}q^4. \tag{12}$$

From the expressions above, it is apparent that the parameter $q \equiv M_2/M^3$ representing the quadrupole moment, enters into the multipole moments M_{2n}, for $n \geq 2$, only at order 2 or higher. Accordingly, solution (10)–(11) for an small value of q, up to order q, may be interpreted as the gravitational field outside a quasi-spherical source. The spacetime being represented by a quadrupole correction to the monopole (Schwarzschild) solution. This is in contrast with other previously mentioned solutions of Weyl family, where all moments higher than the quadrupole are of the same order in q as the quadrupole, and therefore for small
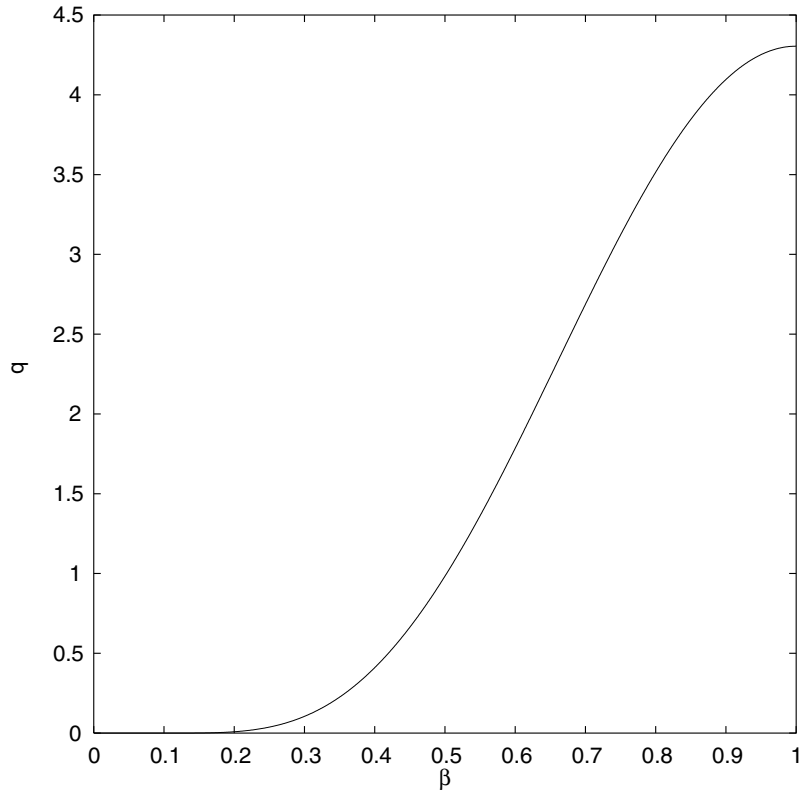
Fig. 1 q (multiplied by 10^4) as a function of β, for $\beta_o = 0.1$, $\beta_s = 1.9$ and $q_o = 10^{-3}$. The maximum value of q_Σ is 4.3046721×10^{-4} which we keep for the exterior region

values of q they cannot be interpreted as a quadrupole perturbation of spherical symmetry.

Instead of cylindrical coordinates (ρ, z), it will be useful for the next section to work with Erez-Rosen coordinates (r, θ) given by:

$$z = (r - M)\cos\theta,$$
$$\rho = (r^2 - 2Mr)^{1/2}\sin\theta, \tag{13}$$

and related to prolate coordinates, by
$$x = \frac{r}{M} - 1,$$
$$y = \cos\theta. \tag{14}$$

The metric functions of the solution, up to the first order in q, hereafter refered as $M\text{--}Q^{(1)}$ are:

$$\Psi_{M-Q}^{(1)} = \frac{1}{2}\ln\left(1 - \frac{2}{R}\right) + \frac{5}{32}q(3y^2 - 1)(3R^2 - 6R + 2)\ln\left(1 - \frac{2}{R}\right)$$
$$- \frac{5}{8}q\ln\left(1 - \frac{2}{R}\right) - \frac{5}{4}q\frac{R-1}{(R-1)^2 - y^2} + \frac{15}{16}q(3y^2 - 1)(R - 1), \quad (15)$$
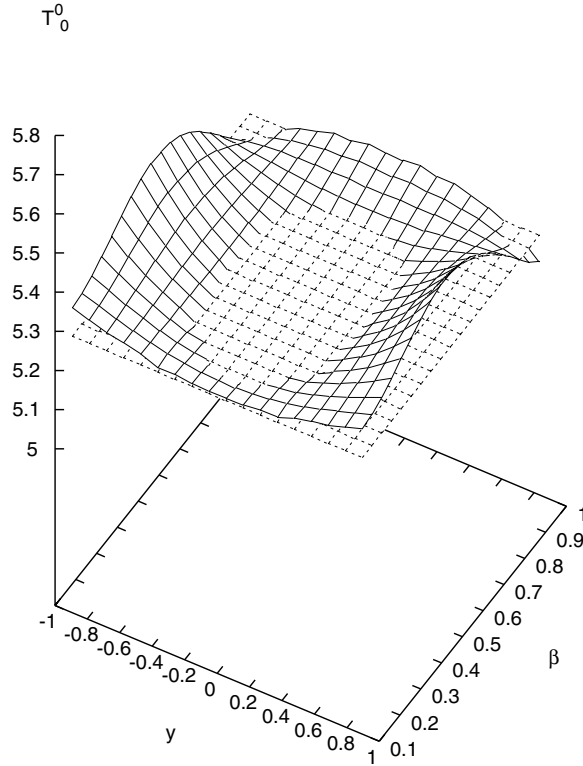
Fig. 2 T_0^0 (multiplied by 10^7) as a function of β and y, for $\alpha = 4/9 \times 10^{-5}$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-12}$

$$\Gamma_{M-Q}^{(1)} = \frac{1}{2} \ln \left[\frac{(R-1)^2 - 1}{(R-1)^2 - y^2} \right] - \frac{15}{8} q (1 - y^2)(R-1) \ln \left(1 - \frac{2}{R} \right)$$

$$- \frac{15}{4} q (1 - y^2) - \frac{5}{4} q (1 - y^2) \left[\frac{(R-1)^2 + y^2}{((R-1)^2 - y^2)^2} \right], \tag{16}$$

with $R \equiv r/M$. An study of the geodesics in this spacetime, has been recently presented [24]

In the next section we shall construct a source for M–$Q^{(1)}$ solution.

3 An interior M–$Q^{(1)}$ metric

The Hernández method [11] is based on a heuristic procedure which allows, starting with some spherically symmetric "seed" source, to obtain interior solutions describing axialsymmetric static sources, which match smoothly on the boundary surface, to a given metric of the Weyl family (see [11] for details). Some applications of this method may be found in [25].
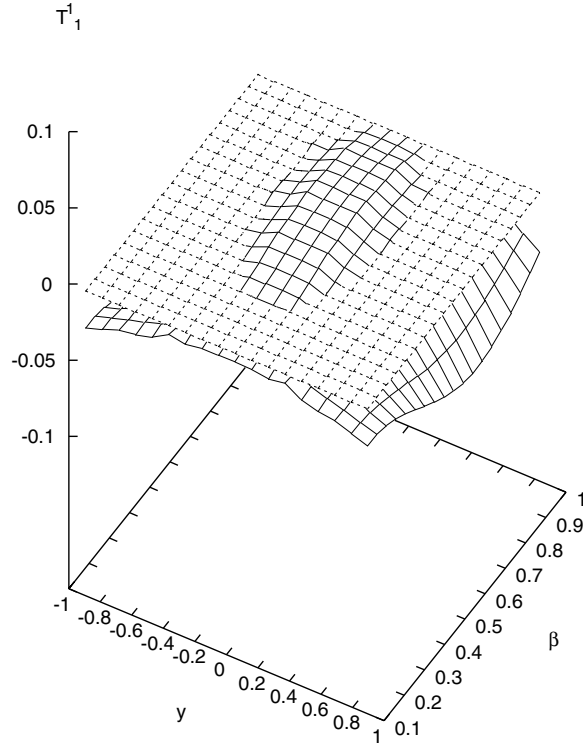
$$T^1_1$$

Fig. 3 T^1_1 (multiplied by 10^7) as a function of β and y, for $\alpha = 4/9 \times 10^{-5}$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-12}$

For our exterior spacetime we shall choose the M–$Q^{(1)}$ solution described above (up to first order in q) and our "seed" interior fluid will be the incompressible Schwarzschild interior solution.

Thus, our exterior metric in Erez-Rosen coordinates read:

$$
\begin{aligned}
g_{rr} &= \mathrm{e}^{2\Gamma - 2\Psi}\left(1 + \frac{\lambda^2 \sin^2\theta}{1 - 2\lambda}\right) = \mathrm{e}^{2q(\Gamma_{q1} - \Psi_{q1})}/(1 - 2M/r), \\
g_{\theta\theta} &= \mathrm{e}^{2\Gamma - 2\Psi}r^2(1 - 2\lambda + \lambda^2 \sin^2\theta) = r^2 \mathrm{e}^{2q(\Gamma_{q1} - \Psi_{q1})}, \\
g_{\phi\phi} &= \mathrm{e}^{-2\Psi}r^2 \sin^2\theta(1 - 2\lambda) = r^2 \sin^2\theta\, \mathrm{e}^{-2q\Psi_{q1}}, \\
g_{tt} &= -\mathrm{e}^{2\Psi} = -(1 - 2M/r)\,\mathrm{e}^{2q\Psi_{q1}},
\end{aligned}
\tag{17}
$$

where the metric functions Γ_{q1} and Ψ_{q1} are given by

$$
\begin{aligned}
\Psi_{q1} = {}& \frac{5}{32}(3y^2 - 1)(3R^2 - 6R + 2)\ln\left(1 - \frac{2}{R}\right) \\
& - \frac{5}{8}\ln\left(1 - \frac{2}{R}\right) - \frac{5}{4}\frac{R - 1}{(R - 1)^2 - y^2} + \frac{15}{16}(3y^2 - 1)(R - 1),
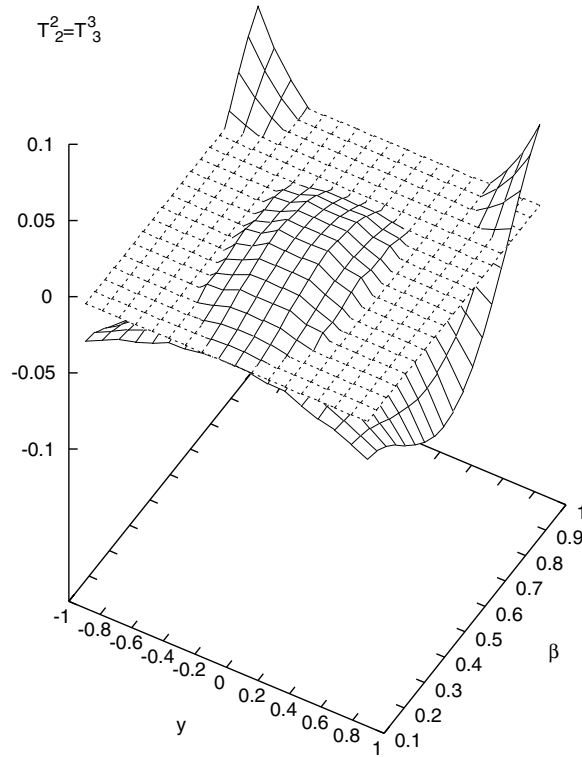\end{aligned}
\tag{18}
$$

Fig. 4 $T_2^2 = T_3^3$ (multiplied by 10^7) as a function of β and y, for $\alpha = 4/9 \times 10^{-5}$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-12}$

$$\Gamma_{q^1} = -\frac{15}{8}(1 - y^2)(R - 1)\ln\left(1 - \frac{2}{R}\right)$$
$$- \frac{15}{4}(1 - y^2) - \frac{5}{4}(1 - y^2)\left[\frac{(R-1)^2 + y^2}{((R-1)^2 - y^2)^2}\right]. \tag{19}$$

Now, Darmois conditions, in these coordinates, imply that metric components as well as $g_{\theta\theta,r}$, $g_{tt,r}$, $g_{\phi\phi,r}$ are continuous across the boundary surface (but allows a jump in $g_{rr,r}$).

Thus, in the example given by Hernández [11], the following substitutions on the chosen exterior metric were applied, in order to obtain the interior metric:

$$2M/r \rightarrow r^2/B^2, \tag{20}$$

for the g_{rr} metric component and the following one for the other metric components

$$2M/r \rightarrow 1 - \left[\frac{3}{2}\left(1 - \frac{r_\Sigma^2}{B^2}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{r^2}{B^2}\right)^{1/2}\right]^2, \tag{21}$$
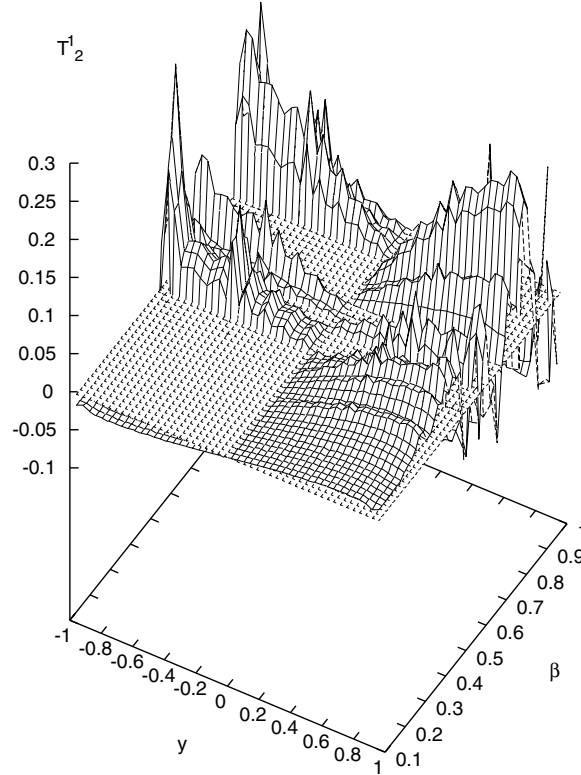
Fig. 5 T_2^1 (multiplied by 10^{19}) as a function of β and y, for $\alpha = 4/9 \times 10^{-5}$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-12}$

where $r = r_\Sigma$ is the equation of the boundary surface of the source and $B^2 = 3/(8\pi\rho_s)$, with ρ_s denoting the energy density of the spherically symmetric "seed" solution. Since (20) does not have a continuous derivative at the surface, but (21) has, it is clear that Darmois conditions will be satisfied by the so obtained metric, on the boundary surface.

However, in our case these substitutions lead to a metric whose g_{rr} component is singular at the origin. Thus as a first step, we are going to modify the Hernández rule, concerning the g_{rr} component. In order to have a regular metric at the origin, we shall use (21) in those terms of g_{rr} which produce the singularity at the origin and (20) in the remaining terms. For the others metric components we apply (21). In addition, we modify the spherical factor in $g_{\theta\theta}$ and $g_{\phi\phi}$ by substituting r^2 with $r^2 + q(r - r_\Sigma)^2$, in order to ensure regularity at the origin (however, as we shall see below this is not enough to assure the correct physical behaviour at the centre). Thus we obtain,

$$g_{tt} = -X(r)^{1+a(r,y)} e^{b(r,y)}, \tag{22}$$

$$g_{rr} = (1 - r^2/B^2)^{-(1+c(r,y))} e^{d(r,y)}, \tag{23}$$

$$g_{\theta\theta} = (r^2 + q(r - r_\Sigma)^2) X(r)^{-c(r,y)} e^{d(r,y)}, \tag{24}$$

$$g_{\phi\phi} = (r^2 + q(r - r_\Sigma)^2) \sin^2\theta \, X(r)^{-a(r,y)} e^{-b(r,y)}, \tag{25}$$
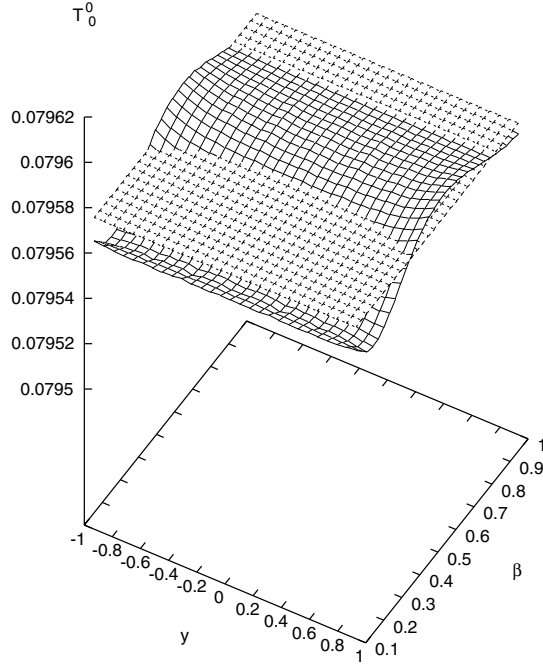
Fig. 6 T_0^0 as a function of β and y, for $\alpha = 6/9$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-4}$

with

$$X(r) \equiv \left[\frac{3}{2} \left(1 - \frac{r_\Sigma^2}{B^2} \right)^{1/2} - \frac{1}{2} \left(1 - \frac{r^2}{B^2} \right)^{1/2} \right]^2, \tag{26}$$

$$a(r, y) = \frac{15}{8} q \frac{-(1 + X(r)^2)(1 - y^2) + 4X(r)y^2}{(1 - X(r))^2}, \tag{27}$$

$$b(x, y) = \frac{5}{2} q \frac{1 + X(r)}{1 - X(r)} \left[\frac{3}{4}(3y^2 - 1) - \frac{(1 - X(r))^2}{(1 + X(r))^2 - y^2(1 - X(r))^2} \right], \tag{28}$$

$$c(r, y) = \frac{15}{16} q \left[4(1 - y^2) \frac{1 + X(r)}{1 - X(r)} + (3y^2 - 1) \frac{(1 + X(r))^2}{(1 - X(r))^2} - y^2 - 1 \right], \tag{29}$$

$$d(r, y) = -\frac{5}{2} q \left[3(1 - y^2) + \frac{3}{4}(3y^2 - 1) \frac{1 + X(r)}{1 - X(r)} \right.$$
$$+ (1 - y^2)(1 - X(r))^2 \frac{(1 + X(r))^2 + y^2(1 - X(r))^2}{[(1 + X(r))^2 - y^2(1 - X(r))^2]^2}$$
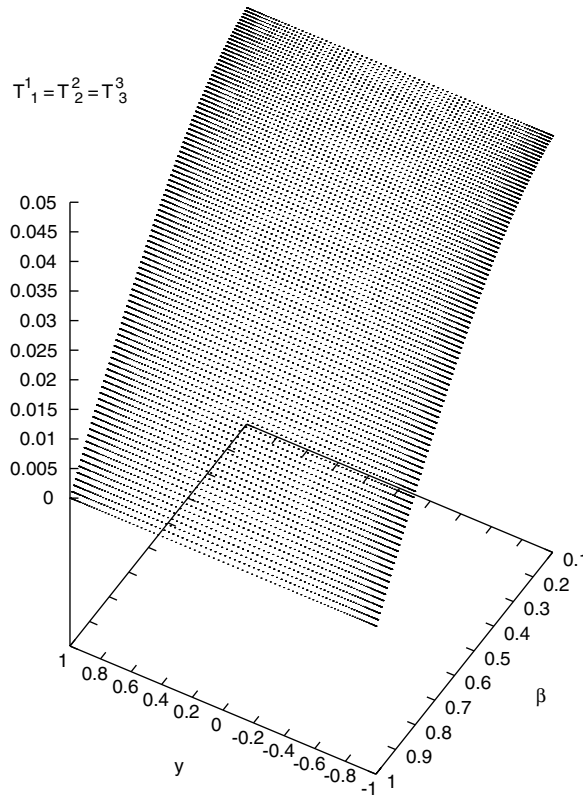$$\left. - \frac{1 - X(r)^2}{(1 + X(r))^2 - y^2(1 - X(r))^2} \right]. \tag{30}$$

Fig. 7 $T_1^1 = T_2^2 = T_3^3$ as a function of β and y, for $\alpha = 6/9$. All the surfaces overlap for $q_o = 0$ and $q_o = -10^{-4}$

Next, let us introduce the following dimensionless variables: $\alpha = 2M/r_\Sigma$ and $\beta = r/r_\Sigma$, in terms of which we may write

$$X(\beta) \equiv \left[\frac{3}{2}(1-\alpha)^{1/2} - \frac{1}{2}(1-\beta^2\alpha)^{1/2}\right]^2. \tag{31}$$

We can now calculate the components of the energy–momentum tensor. However when this is done, negative values of the energy density close to the center appear, even in the weak field limit. To solve this problem and to assure a correct physical behaviour of all physical variables, we shall consider the parameter q as a function of r, such that

$$q = 0; \quad \beta \in [0, \beta_o] \tag{32}$$

and

$$q = q_o(\beta - \beta_o)^4(\beta - \beta_s)^4; \quad \beta \in [\beta_o, (\beta_o + \beta_s)/2], \tag{33}$$

where β_o, β_s and q_o are constants such that $\beta_o + \beta_s = 2$ and the value of q at the boundary surface, which is $q_o[(\beta_o - \beta_s)/2]^8$ coincides with the quadrupole parameter of the exterior M–$Q^{(1)}$ solution, i.e. $q = q_\Sigma$, where q_Σ denotes the
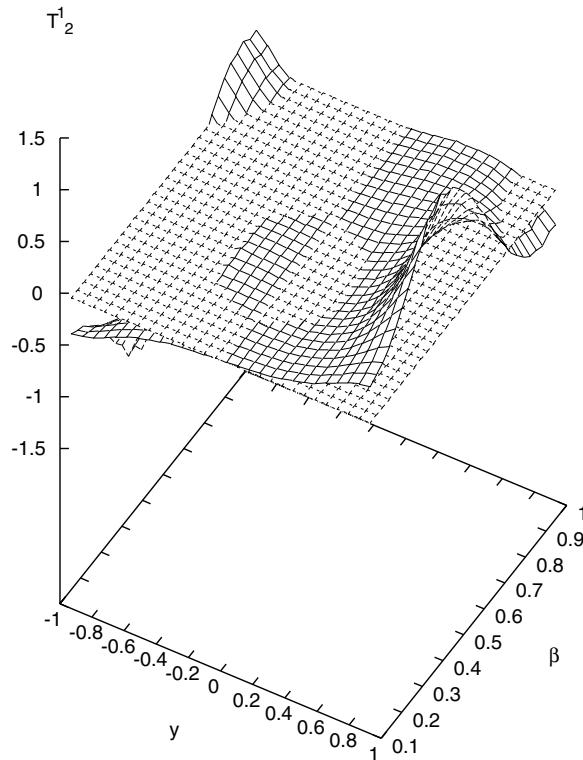
Fig. 8 T_2^1 (multiplied by 10^6) as a function of β and y, for $\alpha = 6/9$. Dashed lines indicate $q_o = 0$ and continuous lines $q_o = -10^{-4}$

quadrupole parameter of the exterior $M–Q^{(1)}$. It should be observed that since both q and its first derivative are continuous across the boundary surface, junction conditions are satisfied, after the replacement above. The specific form of β as well as the values of different parameters, are indicated in Fig.1 and in its legend.

Thus our source consists of a spherical inner core continuously matched to a non-spherical distribution of matter, producing a $M–Q^{(1)}$ spacetime at the outside, and satisfying the continuity of the first and the second fundamental forms at the boundary surface.

Calculations of different physical variables show their correct physical behaviour within the matter distribution.

Expressions are extremely lengthy and therefore we omit them here, however they are available upon request to W. Barreto.

The non-vanishing components are $T_0^0, T_1^1, T_2^2, T_3^3, T_2^1$. For $\alpha < 0.7$ all these components are regular within the fluid distribution and energy conditions are satisfied (e.g. energy density is positive and larger than stresses). In the weak field limit ($\alpha \ll 1$) we obtain a quadrupole correction to the incompressible fluid.

4 Conclusions

We have seen that the M–$Q^{(1)}$ solution satisfies the requested condition to be considered as a quadrupole correction to the spherical symmetry, namely: all relativistic moments higher than the quadrupole are of higher order in q. Accordingly it represents, among the known members of the Weyl family, a good candidate to describe small departures from spherical symmetry.

Next, we have found a source for that space-time. The interior solution obtained by an application of the Hernández algorithm, matches smoothly to the M–$Q^{(1)}$ metric on the boundary of the matter distribution, is regular and satisfies all standard physical conditions. In the weak field approximation it represents a quadrupole correction to the incompressible fluid sphere.

As we have mentioned before, a spherical core was introduced in order to assure acceptable physical behaviour at the centre. It consists of a sphere of incompressible fluid with positive energy density, larger than pressure and which matches smoothly to the outer part of the source. For the models presented below the spherical core represents about the 0.1% of the total volume of the source.

Since we are interested in slight deviations from spherical symmetry we shall consider models with very small values of q. Indeed the magnitude of the q's in the models presented are many orders of magnitude smaller than the values corresponding to the earth and the sun, which in our units are approximately 10^{15} and 10^5 respectively [9]. However for a neutron star of one solar mass, 10 km radius and the same eccentricity as the sun, the order of magnitude of q is 10^{-4} [9], as in one of the models below.

It is also worth noticing that the critical value of alpha (≈ 0.7) is smaller than the corresponding value in the exactly spherically symmetric case (8/9)

We have ran a large number of models for a wide range of values of the parameters and, both, positive and negative q's. Below we show figures corresponding to two models, one with small α and the other with a value of α close to its critical value. In both models we considered negative values of q since we are primarily interested in oblate objects.

Figs. 2–5 exhibit the behaviour of physical variables for an small value of α (weak field limit). Despite the small value of the quadrupole parameter, its contribution is clearly shown in Figs. 2–4. In Fig. 5 the numerical error is comparable with the quadrupole contribution, and accordingly the latter is somehow screened by the former.

Figures 6–8 display the physical variables for a large value of α. In this case, the quadrupole correction appears sharply in T_0^0 and T_2^1. However it is negligible in T_1^1, T_2^2 and T_3^3.

Finally, Fig. 9 display the range of parameters α and q for which the solution is physically acceptable.

It is our hope that this source as well as other interiors to some Weyl metrics presented elsewhere ([25] and [26]) could be used as initial configurations to describe the departure from equilibrium of very compact objects endowed with an small but non-vanishing quadrupole structure.
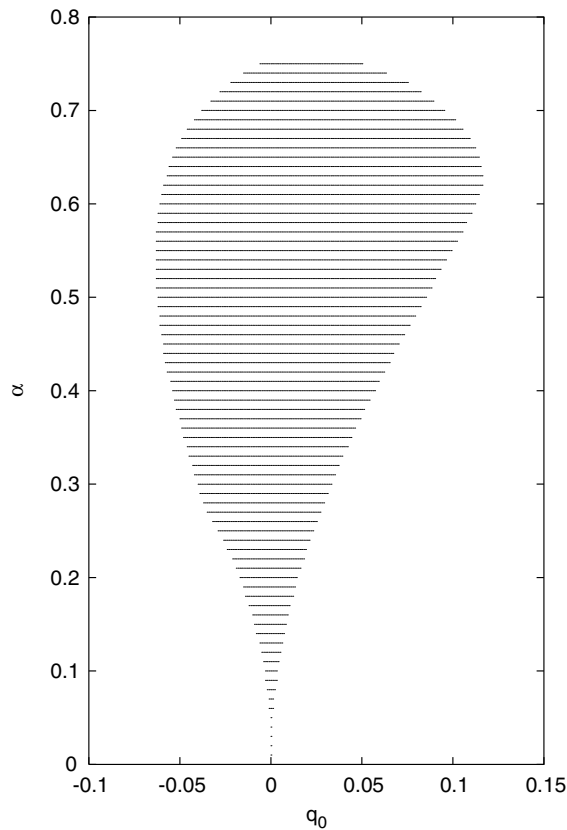
Fig. 9 Dashed region indicates the range of α and q_o for which models satisfy $T_0^0 > 0$ and $T_0^0 > T_1^1$

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