Thermoinertial bouncing of a relativistic collapsing sphere: A numerical model

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We present a numerical model of a collapsing radiating sphere, whose boundary surface undergoes bouncing due to a decreasing of its inertial mass density (and, as expected from the equivalence principle, also of the "gravitational" force term) produced by the "inertial" term of the transport equation. This model exhibits for the first time the consequences of such an effect, and shows that under physically reasonable conditions this decreasing of the gravitational term in the dynamic equation may be large enough as to revert the collapse and produce a bouncing of the boundary surface of the sphere.

I. INTRODUCTION

In the study of gravitational collapse of massive stars, the inclusion of dissipative processes (in particular neutrino emission) is enforced by the fact that they provide the only plausible mechanism to carry away the bulk of binding energy, leading to a neutron star or black hole [1]. On the other hand, in cores of densities about 10^{12} g cm^{-3} the mean free path of neutrinos becomes small enough as to justify the use of diffusion approximation [2,3]. This seems to be confirmed by the observational data collected from supernova 1987A, which indicates that the radiation transport regime prevailing during the emission process, is closer to the diffusion approximation than to the streaming out limit [4].

Motivated by the comments above, in a recent paper [5], the Misner and Sharp approach to the study of adiabatic gravitational collapse [6] was extended as to include dissipation in, both, the streaming out and diffusion approximation (for the case of pure free streaming approximation see [7]). Then from the coupling of the dynamical equation to a causal transport equation in the context of Müller-Israel-Stewart theory [8,9] it was obtained that the effective inertial mass density of a fluid element and the gravitational force term in the dynamical equation, reduce by a factor which depends on dissipative variables. This reduction, in its turn, might lead to the bouncing of the collapsing sphere, as discussed in [5].

As can be seen from inspection of the transport equation, such an effect is directly related to the presence of the inertial term Ta_β in the transport equation. This explains why we refer to such a bouncing as "thermo-inertial".

It is our purpose in this work to present a numerical model of a radiating collapsing sphere, where the above

mentioned effect produces the bouncing of the boundary surface of the sphere, for physically acceptable values of all variables.

Since we are mainly concerned with time scales of the order of magnitude of (or even smaller than) the hydrostatic time scale, as in the quick collapse phase preceding neutron star formation, we cannot rely on the quasistatic approximation, and therefore the full dynamic description has to be used [10,11]. This implies that we have to appeal to a hyperbolic theory of dissipation. The use of a hyperbolic theory of dissipation is further justified by the necessity of overcoming the difficulties inherent to parabolic theories (see Refs. [12–24] and references therein).

The plan of the paper is as follows. In the next section we define the conventions and present the dynamical equation coupled to the transport equation. The model to be considered as well as the strategy for the numerical integration is presented in Sec. III. Finally, a discussion of results is presented in Sec. IV.

II. THE DYNAMICAL EQUATION OF THE DISSIPATIVE FLUID

We consider a spherically symmetric distribution of collapsing fluid (for simplicity we shall consider the pressure to be locally isotropic) undergoing dissipation in the form of heat flow, bounded by a spherical surface Σ. We assume the interior metric to Σ to be comoving, shear-free for simplicity, and spherically symmetric, accordingly it may be written as

$$ds^2 = -A^2(t, r)dt^2 + B^2(t, r)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2),$$
(1)

and hence we have for the four velocity V^α and the heat flux vector q^α

$$V^\alpha = A^{-1}\delta_0^\alpha, \qquad q^\alpha = q\delta_1^\alpha.$$
(2)

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Then it can be shown [5] that the following equation can be found from Bianchi identities

$$(\mu + P)D_t U = -(\mu + P)[m + 4\pi PR^3]\frac{1}{R^2} - E^2 D_R P$$
$$- E\left[5qB\frac{U}{R} + BD_t q\right], \tag{3}$$

where μ is the energy density, P the pressure,

$$D_t = \frac{1}{A}\frac{\partial}{\partial t}, \tag{4}$$

the proper radial derivative D_R, constructed from the radius of a spherical surface, as measured from its perimeter inside Σ, being

$$D_R = \frac{1}{R'}\frac{\partial}{\partial r}, \tag{5}$$

with

$$R = rB, \tag{6}$$

and where dots and primes denote derivatives with respect to t and r respectively. The velocity U of the collapsing fluid is defined as

$$U = rD_t B < 0 \quad \text{(in the case of collapse)}. \tag{7}$$

Also, the mass function $m(t, r)$ of Cahill and McVittie [24] is obtained from the Riemann tensor component $R_{23}{}^{23}$ and is for metric (1)

$$m(t, r) = \frac{(rB)^3}{2}R_{23}{}^{23} = \frac{r^3}{2}\frac{B\dot{B}^2}{A^2} - \frac{r^3}{2}\frac{B'^2}{B} - r^2 B', \tag{8}$$

E is defined as

$$E = \frac{(rB)'}{B} = \left[1 + U^2 - \frac{2m(t, r)}{rB}\right]^{1/2}. \tag{9}$$

Next, the corresponding transport equation for the heat flux reads [8,9]

$$\tau h^{\alpha\beta}V^\gamma q_{\beta;\gamma} + q^\alpha = -\kappa h^{\alpha\beta}(T_{,\beta} + Ta_\beta)$$
$$- \frac{1}{2}\kappa T^2\left(\frac{\tau V^\beta}{\kappa T^2}\right)_{;\beta} q^\alpha, \tag{10}$$

where $h^{\mu\nu}$ is the projector onto the three space orthogonal to V^μ, κ denotes the thermal conductivity, and T and τ denote temperature and relaxation time, respectively. Observe that due to the symmetry of the problem, Eq. (10) only has one independent component, which may be written as:

$$BD_t q = -\frac{\kappa T}{\tau E}D_t U - \frac{\kappa T'}{\tau B} - \frac{qB}{\tau}\left(1 + \frac{\tau U}{R}\right)$$
$$- \frac{\kappa T}{\tau E}[m + 4\pi P]R^{-2} - \frac{\kappa T^2 qB}{2A\tau}\left(\frac{\tau}{\kappa T^2}\right) - \frac{3UBq}{2R}, \tag{11}$$

Then coupling (3) to (11) one obtains (some misprints in Eq. (39) in [5] has been corrected here)

$$(\mu + P)(1 - \alpha)D_t U = F_{\text{grav}}(1 - \alpha) + F_{\text{hyd}} + \frac{E\kappa T'}{\tau B}$$
$$+ \frac{EqB}{\tau} - \frac{5qBEU}{2R}$$
$$+ \frac{\kappa ET^2 qB}{2A\tau}\left(\frac{\tau}{\kappa T^2}\right), \tag{12}$$

where F_{grav} and F_{hyd} are defined by

$$F_{\text{grav}} = -(\mu + P)[m + 4\pi PR^3]\frac{1}{R^2}, \tag{13}$$

and

$$F_{\text{hyd}} = -E^2 D_R P, \tag{14}$$

with α given by

$$\alpha = \frac{\kappa T}{\tau(\mu + P)}. \tag{15}$$

Thus as α tends to 1, the effective inertial mass density of the fluid element tends to zero. Furthermore observe that F_{grav} is also multiplied by the factor $(1 - \alpha)$. Indicating that the effective gravitational attraction on any fluid element decreases by the same factor as the effective inertial mass (density). This of course is to be expected, from the equivalence principle. It is also worth mentioning that F_{hyd} is in principle independent (at least explicitly) on this factor.

With these last comments in mind, let us now imagine the following situation. As far as the right hand side of (12) is negative, the system keeps collapsing. However, let us assume that the collapsing sphere evolves in such a way that, for some region of the sphere, the value of α increases and approaches the critical value of 1. Then, as this process goes on, the ensuing decreasing of the gravitational force term would eventually lead to a change of the sign of the right hand side of (12). Since that would happen for small values of the effective inertial mass density, that would imply a strong bouncing of that part of the sphere, even for a small absolute value of the right hand side of (12).

In the next section a model will be presented where the effect above appears explicitly. For simplicity we shall consider a particular case of the transport equation, corresponding to the so called truncated version, in which case the last term on the right of (10) is absent [14,20]. In this case, (12) becomes

$$(\mu + P)(1 - \alpha)D_t U = F_{\text{grav}}(1 - \alpha) + F_{\text{hyd}} + \frac{E\kappa T'}{\tau B}$$
$$+ \frac{EqB}{\tau} - \frac{4qBEU}{R}. \tag{16}$$

III. THE MODEL

In this section we shall present a numerical model where the decreasing of the effective mass mentioned in the previous section will produce a bouncing during the evolution of a dissipative sphere. For simplicity we shall assume our fluid to be shear-free and conformally flat, and also that a relevant increase of α takes place only at the boundary surface of the sphere. Thus we shall need only to integrate at the boundary surface, implying that we shall deal with ordinary differential equations for variables defined on that surface.

A. The general form of the metric and the field equations

If the fluid sphere is shear-free and conformally flat, the metric functions take the form [25]

$$A = [C_1(t)r^2 + 1]B \tag{17}$$

and

$$B = \frac{1}{C_2(t)r^2 + C_3(t)}, \tag{18}$$

where C_1, C_2 and C_3 are arbitrary functions of t. Although the shear-free and the conformally flat conditions are introduced here in order to simplify calculations, it is worth noticing that these conditions generalize physical assumptions widely used in astrophysics. Indeed, the shear-free condition in the Newtonian regime describes the homologous evolution and has been extensively considered in

general relativity [26]. On the other hand it is well known that the conformally flat condition implies in the perfect fluid case the homogeneity of the energy density distribution.

For the numerical integration we shall need to write all variables in dimensionless form, accordingly we shall redefine the metric functions C_1 and C_2 by (C_3 is already dimensionless):

$$C_{1,2} \rightarrow \frac{C_{1,2}}{r_\Sigma^2}, \tag{19}$$

where $r = r_\Sigma = $ constant defines the boundary surface of the fluid sphere.

In terms of these dimensionless functions, A and B become

$$A = [C_1(t)(r/r_\Sigma)^2 + 1]B \tag{20}$$

and

$$B = \frac{1}{C_2(t)(r/r_\Sigma)^2 + C_3(t)}. \tag{21}$$

Then the following expressions for the physical variables are obtained from Einstein equations

$$\mu r_\Sigma^2 = \frac{3}{8\pi}\left(\frac{\dot{C}_2(r/r_\Sigma)^2 + \dot{C}_3}{C_1(r/r_\Sigma)^2 + 1}\right)^2 + \frac{3}{2\pi}C_2C_3, \tag{22}$$

$$r_\Sigma^2 P = \frac{1}{8\pi(C_1(r/r_\Sigma)^2 + 1)^2}[2(\ddot{C}_2(r/r_\Sigma)^2 + \ddot{C}_3)(C_2(r/r_\Sigma)^2 + C_3) - 3(\dot{C}_2(r/r_\Sigma)^2 + \dot{C}_3)^2$$
$$- 2\frac{\dot{C}_1(r/r_\Sigma)^2}{C_1(r/r_\Sigma)^2 + 1}(\dot{C}_2(r/r_\Sigma)^2 + \dot{C}_3)(C_2(r/r_\Sigma)^2 + C_3)] + \frac{1}{2\pi(C_1(r/r_\Sigma)^2 + 1)}$$
$$\times [C_2(C_2 - 2C_1C_3)(r/r_\Sigma)^2 + C_3(C_1C_3 - 2C_2)], \tag{23}$$

$$qr_\Sigma^2 = \frac{1}{2\pi}(r/r_\Sigma)(\dot{C}_3C_1 - \dot{C}_2)\left(\frac{C_2(r/r_\Sigma)^2 + C_3}{C_1(r/r_\Sigma)^2 + 1}\right)^2, \tag{24}$$

where from now on dot denotes derivative with respect to t/r_Σ.

B. The surface equations

Next, from (6) we obtain for the dimensionless proper radius of the sphere

$$R_\Sigma = \frac{1}{C_2 + C_3}, \tag{25}$$

and from (9) evaluated at the boundary surface

$$E_\Sigma = \frac{C_3 - C_2}{C_3 + C_2}. \tag{26}$$

Solving these two equations we obtain

$$C_2 = \frac{1 - E_\Sigma}{2R_\Sigma} \tag{27}$$

and

$$C_3 = \frac{1 + E_\Sigma}{2R_\Sigma}. \tag{28}$$

On the other hand we have

$$A_\Sigma = (C_1 + 1)R_\Sigma, \tag{29}$$

and from (7)

$$U_\Sigma = -\frac{\dot{C}_2 + \dot{C}_3}{(C_1 + 1)(C_2 + C_3)}, \qquad (30)$$

where, again, dots denote derivatives with respect to the dimensionless time t/r_Σ. Using (25) and (29) in (30) we may write

$$\dot{R}_\Sigma = A_\Sigma U_\Sigma, \qquad (31)$$

which is our first equation at the surface.

Next, we use the total loss of mass equation which can be easily derived from (8) and the junction condition

$$P_\Sigma = (qB)_\Sigma, \qquad (32)$$

to obtain (see [5]) for details)

$$\dot{M}_\Sigma = -Q(t) A_\Sigma R_\Sigma (U_\Sigma + E_\Sigma), \qquad (33)$$

where M_Σ is the dimensionless mass, $Q(t) \equiv 4\pi q_\Sigma R_\Sigma^2$ and q_Σ denotes the dimensionless heat flow qr_Σ^2, evaluated at the boundary surface. This is our second surface equation.

Finally, in order to obtain the third surface equation we proceed as follows. From the Eqs. 23 and 32 in [25], it can be shown that

$$\dot{U}_\Sigma = \frac{1}{2R_\Sigma}[A_\Sigma(3E_\Sigma^2 - 1 - U_\Sigma^2 - 2R_\Sigma Q(t)) + 2E_\Sigma(A_\Sigma - 2R_\Sigma)]. \qquad (34)$$

This is the third equation to be integrated at the surface of the distribution.

Thus we have a system of three Eqs. (31), (33), and (34) for the five unknown functions of time R_Σ, A_Σ, U_Σ, E_Σ and Q. In order to integrate such a system, we shall prescribe the "luminosity" (Q), and obtain a constraint equation from (16), on what we shall elaborate as follows .

From the dynamic equation (16), using the boundary condition (32) we obtain the pressure gradient at the surface

$$P'_\Sigma = -\frac{R_\Sigma}{E_\Sigma}\{[1 - \alpha_\Sigma][\mu_\Sigma + P_\Sigma][4\pi R_\Sigma(\mu_\Sigma/3 + P_\Sigma) + \dot{U}_\Sigma/A_\Sigma] - \mathcal{T}_\Sigma\}, \qquad (35)$$

where primes and dots denote derivatives with respect to the dimensionless variables r/r_Σ and t/r_Σ respectively,

$$\mu_\Sigma = \frac{3M_\Sigma}{4\pi R_\Sigma^3}, \qquad (36)$$

$$P_\Sigma = \frac{Q(t)}{4\pi R_\Sigma}, \qquad (37)$$

(observe that μ_Σ and P_Σ denote the dimensionless expressions for the energy density and pressure evaluated at the boundary surface, i.e. these variables multiplied by r_Σ^2) and

$$\mathcal{T}_\Sigma = \alpha_\Sigma(\mu'_\Sigma + P'_\Sigma)\frac{E_\Sigma}{R_\Sigma} + \frac{E_\Sigma Q(t)}{4\pi R_\Sigma \tau} - \frac{4QE_\Sigma U_\Sigma}{4\pi R_\Sigma^2}, \qquad (38)$$

where we have used $kT = \alpha\tau(\mu + P)$ (conveniently adimensionalized) and have assumed for simplicity that $\alpha'_\Sigma = 0$. Finally, using the field equations (22)–(24) we obtain the following expression for A_Σ

$$A_\Sigma = \frac{\tau R_\Sigma[4\alpha_\Sigma(3M_\Sigma/R_\Sigma + QR_\Sigma) - \dot{Q}R_\Sigma(1 + \alpha_\Sigma)]}{2\tau\alpha_\Sigma(1 + E_\Sigma)(QR_\Sigma + 3M_\Sigma/R_\Sigma) - QR_\Sigma(R_\Sigma - 7U_\Sigma\tau)}. \qquad (39)$$

C. Strategy of integration

The integration scheme is now an easy shot: Giving initial conditions for R_Σ, U_Σ and M_Σ, and prescribing α_Σ and $Q(t)$, we can integrate numerically, Eqs. (31), (33), and (34), with the constraint equation (39).

The form of α_Σ is suggested by the very idea underlying the motivation of this work, namely: the fact that as α increases, the ensuing reduction of the gravitational term in the dynamical equation may lead to a bouncing of the sphere. Accordingly, we shall take for α_Σ a smooth function of time, rising from zero to some value below the critical one ($\alpha = 1$).

D. Model

We have ran a large number of models exhibiting bouncing, under physically reasonable conditions, corresponding to a wide range of initial data and values of the parameters, and very different choices of $Q(t)$ and $\alpha(t)$. For all these choices, the qualitative behavior associated to the increase of α is essentially the same. From them we have selected the following model.

The initial conditions are

$$R_\Sigma(0) = 20,$$

$$M_\Sigma(0) = 1,$$

$$U_\Sigma(0) = -0.1,$$

with $\tau = 0.1$. These values correspond to a sphere with an initial radius of the order of 400 Km, an initial mass of the order of 10 solar masses and a relaxation time of the order of 10^{-4} seconds.

The sphere is assumed to be radiating according to

$$Q(t) = Q_0 e^{-(t - t_m)^2/\sigma},$$

where $Q_0 = 0.001$, $t_m = 0.5$ and $\sigma = 0.005$, producing a total mass ejection of the order of 0.1%.

For α we choose

$$\alpha(t) = \alpha_m/(e^{-(t-t_m)/\sigma} + 1),$$

with α_m ranging from 0 to 1.

IV. DISCUSSION

The influence of prerelaxation effects on gravitational collapse has been brought out in many works in last decade [27], however the specific effect of bouncing, associated with the decreasing of the effective inertial mass density, produced by the increasing of α, had not been illustrated until now. It is worth stressing that α-terms in Eq. (12) come from the inertial factor $T a_\beta$ in Eq. (10).

In this work we provide a numerical model of such bouncing, by assuming an increasing of α at the boundary surface. We have concentrated the increase of α on the boundary surface to illustrate the effect, the remaining of the sphere is assumed to be dissipating at much lower values of α. Of course, the increasing of α may in principle occur at any region of the sphere and even in more that one, simultaneously. The results of our integration is deployed in the Figs. 1–6, which exhibit the evolution of different variables with respect to the dimensionless time t/r_Σ.

Figure 1 shows the evolution of R_Σ for different values of α_m from 0 to 1, the bouncing is clearly exhibited as well

FIG. 2. Evolution of R_Σ (continuous line) and α (dashed line) for $\alpha_m = 0.5$. The curves were normalized (and shifted only for R_Σ) in order to display them together.
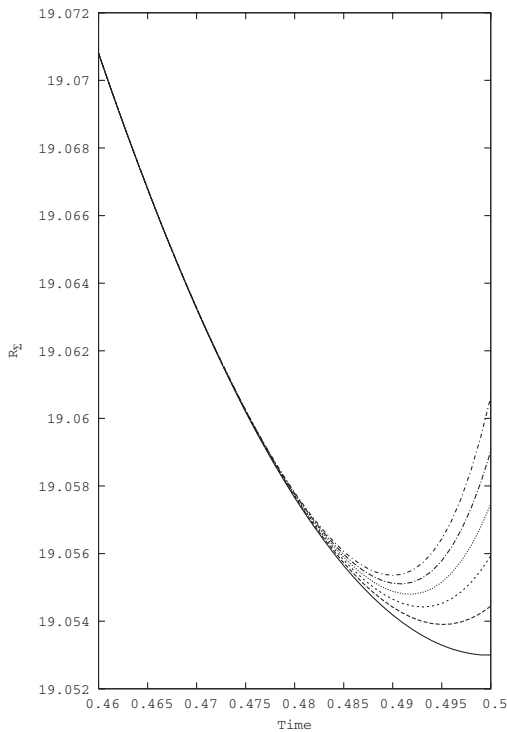

FIG. 1. Evolution of R_Σ for different values of α_m: 0.0 (solid line); 0.2 (large dashed line); 0.4 (short dashed line); 0.6 (dotted line); 0.8 (dot–large dashed line) and 1 (dot–short dashed line).
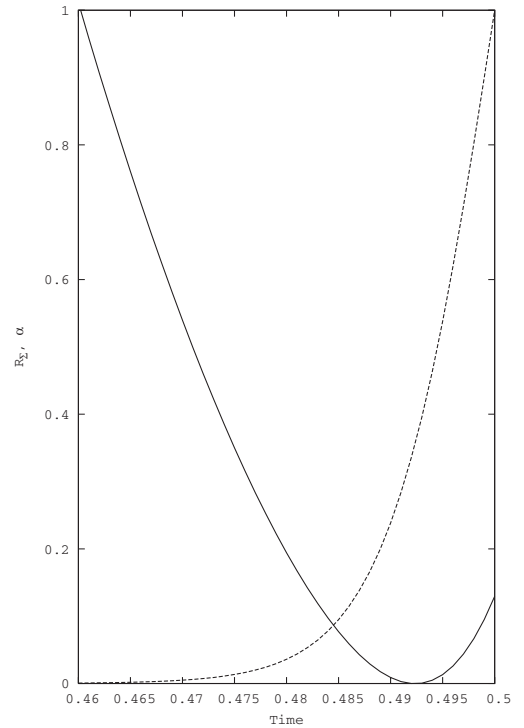
as its dependence on α. Figure 2 emphasizes further the link between the increasing of α and the bouncing.

Figs. 3–6 show the behavior of (dimensionless) energy density, pressure, heat flow and temperature, evaluated at the boundary surface. Their values are always regular and satisfy the physical conditions $\rho > P > 0$.

The dimensionless quantity κT_Σ plotted in Fig. 6 is, in conventional units,

$$\kappa T_\Sigma = 2 10^6 \frac{G}{c^5} [\kappa][T_\Sigma] \qquad (40)$$

with G and c denoting the gravitational constant and the speed of light, and where $[\kappa]$ and $[T]$ denote the numerical values of conductivity and temperature in g cm^{-3} K^{-1} and K respectively. Therefore the maximum values of κT_Σ reached just after the bouncing, correspond to

$$[\kappa][T_\Sigma] \approx 10^{46} \qquad (41)$$

which may be obtained with $[T_\Sigma] \approx 10^{12}$ and $[\kappa] \approx 10^{34}$. These values are well within the acceptable range for those variables in a presupernovae event [28].

Thus we have seen that a relatively simple model, whose physical variables exhibit good behavior and have acceptable numerical values, may serve to illustrate the bouncing of a dissipating self-gravitating sphere, produced by the decreasing of its effective inertial mass density associated to an increasing of α.
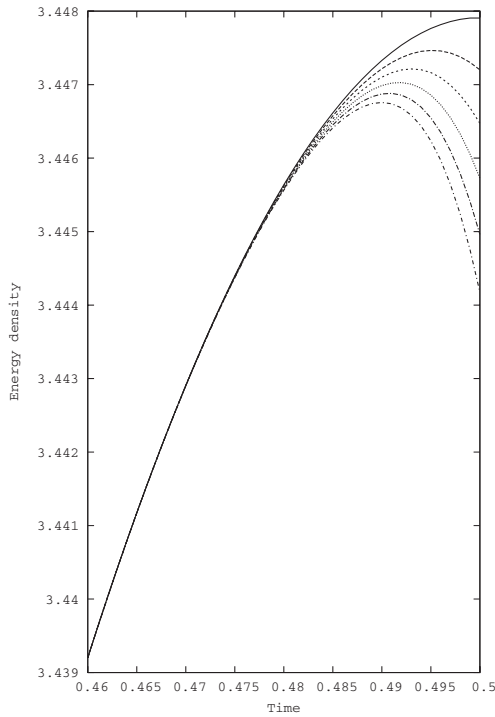
FIG. 3.   Energy density at the surface (multiplied by 10^5) evolution for different values of α_m: 0.0 (solid line); 0.2 (large dashed line); 0.4 (short dashed line); 0.6 (dotted line); 0.8 (dot-large dashed line) and 1 (dot-short dashed line).
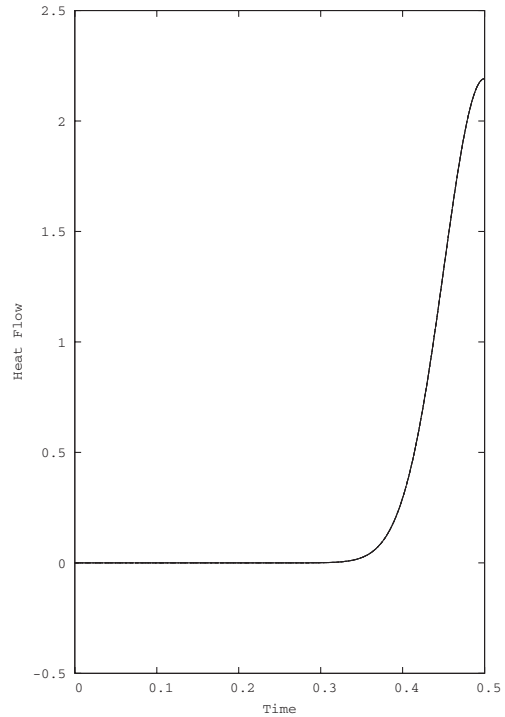

FIG. 5.   Heat flow at the surface (multiplied by 10^7) evolution for the same values of α_m. They all overlap within the approximation of the plotter.
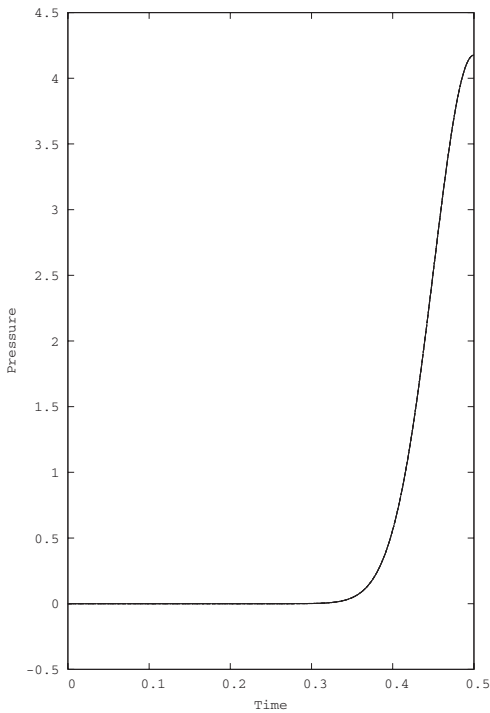

FIG. 4.   Pressure at the surface (multiplied by 10^6) evolution for the same values of α as in previous figure. They all overlap within the approximation of the plotter.
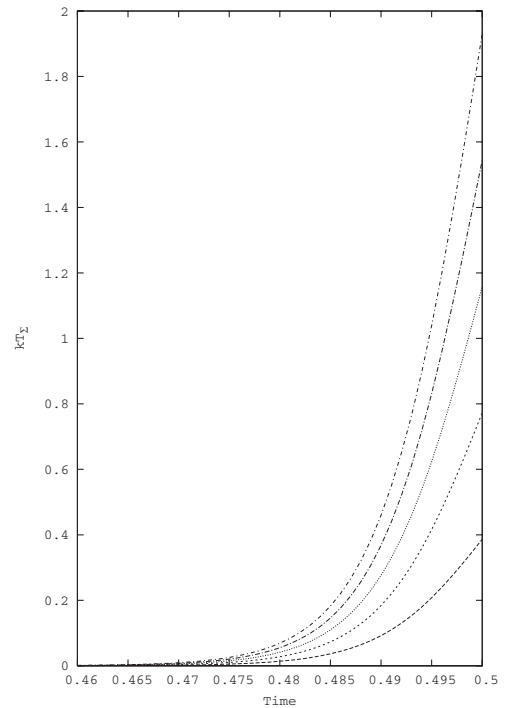

FIG. 6.   Temperature at the surface (multiplied by $\kappa 10^7$) evolution for different values of α_m: 0.0 (solid line); 0.2 (large dashed line); 0.4 (short dashed line); 0.6 (dotted line); 0.8 (dot-large dashed line) and 1 (dot-short dashed line).

Nevertheless, in spite of the appeal of the presented model, we are well aware that invoking such an effect to describe a specific observed phenomena, would require a much more detailed astrophysical setting. This, however, is out of the scope of this paper.

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