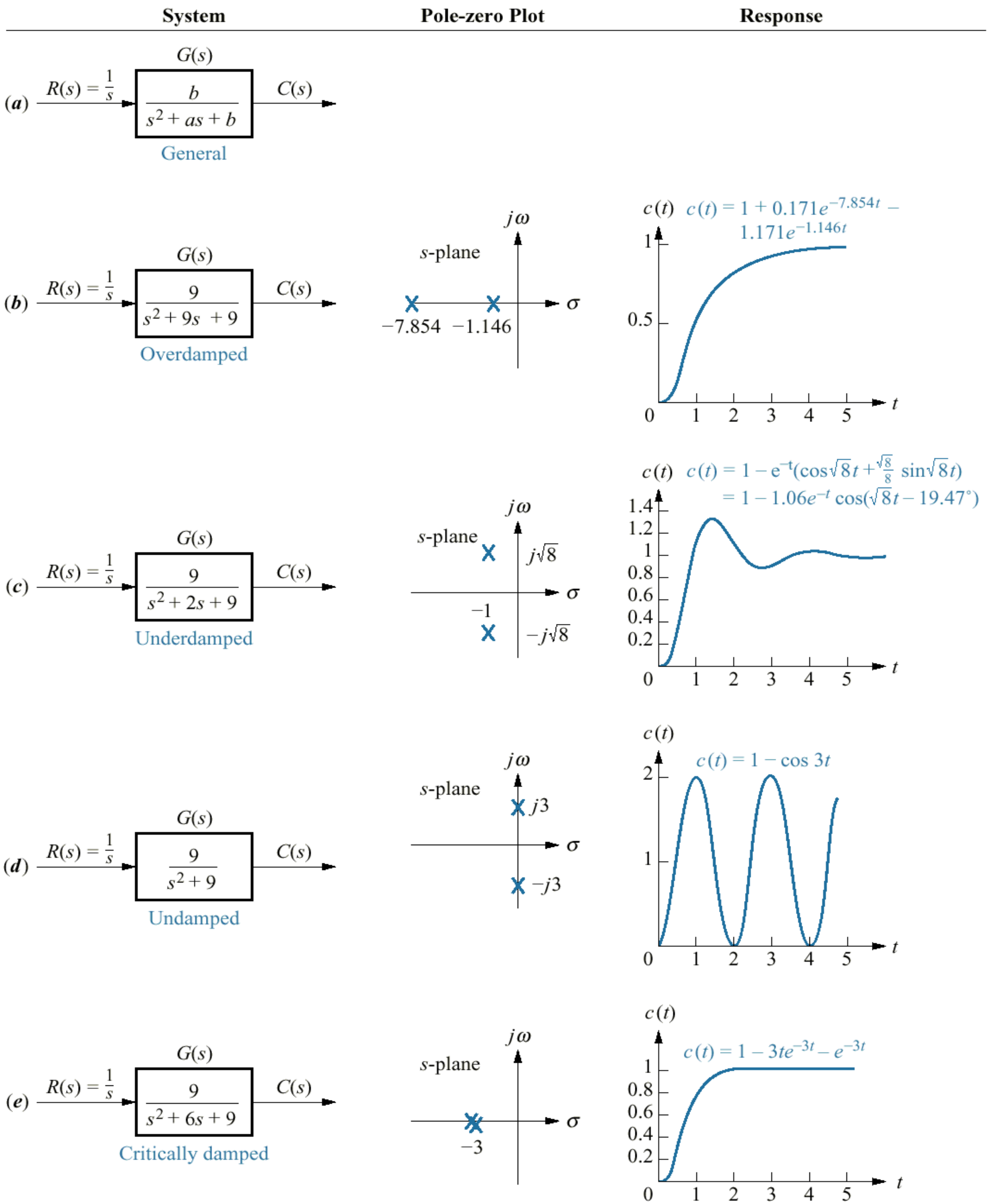
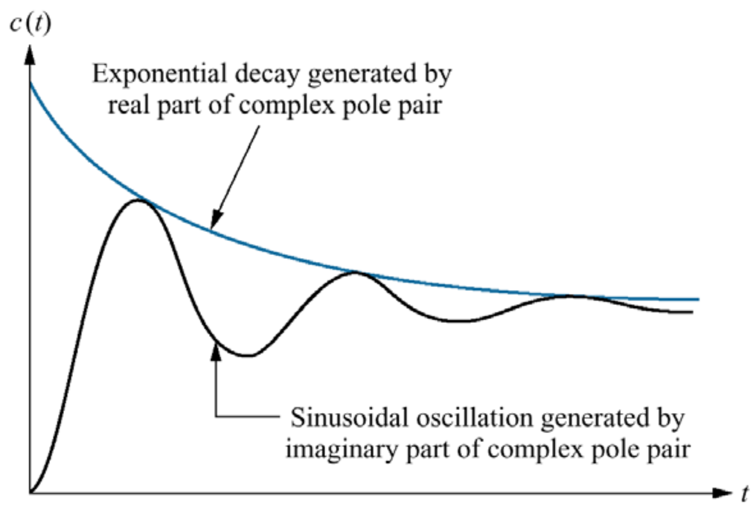

System
Pole-zero Plot
Response
(a) $R(s) = \frac{1}{s}$ $G(s)$ $\frac{b}{s^2 + as + b}$ $C(s)$
General
(b) $R(s) = \frac{1}{s}$ $G(s)$ $\frac{9}{s^2 + 9s + 9}$ $C(s)$
Overdamped
s-plane
$j\omega$
σ
−7.854 −1.146
$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
(c) $R(s) = \frac{1}{s}$ $G(s)$ $\frac{9}{s^2 + 2s + 9}$ $C(s)$
Underdamped
$j\sqrt{8}$
$-j\sqrt{8}$
−1
$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t) = 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)$
(d) $R(s) = \frac{1}{s}$ $G(s)$ $\frac{9}{s^2 + 9}$ $C(s)$
Undamped
$j3$
$-j3$
$c(t) = 1 - \cos 3t$
(e) $R(s) = \frac{1}{s}$ $G(s)$ $\frac{9}{s^2 + 6s + 9}$ $C(s)$
Critically damped
−3
$c(t) = 1 - 3te^{-3t} - e^{-3t}$

- **1. Overdamped response:**
  Poles: Two real and different real part
  Natural response: Two exponentials with time constants equal to the reciprocal of the pole location
- **2. Underdamped responses:**
  Poles: Two complex with real part non zero
  Natural response: Damped sinusoid with and exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles

- **3. Undamped response:**
  Poles: Two imaginary (the real part is zero)
  Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles
- **4. Critically damped responses:**
  Poles: Two real and equals.
  Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time and an exponential with time constant equal to the reciprocal of the pole location



Second-order step response components generated by complex poles

- **Natural Frequency**: The natural frequency of a second-order system is the frequency of oscillation of the system without damping.
- **Damping Ratio**: The damping ratio is defined as the ratio of exponential decay frequency to natural frequency.

Consider the general system: $$G(s) = \frac{b}{s^2 + as + b}$$

Without damping,

$$G(s) = \frac{b}{s^2 + b} \quad \Rightarrow \quad \omega_n = \sqrt{b}$$

$$s_1 = -\frac{a}{2} + j\frac{\sqrt{4b - a^2}}{2}, \quad s_2 = -\frac{a}{2} - j\frac{\sqrt{4b - a^2}}{2}$$

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency\ (rad/\sec)} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\sqrt{b}}$$

$$\Rightarrow \quad a = 2\zeta\omega_n \quad \Rightarrow \quad G(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Step response

$$C(s) = \frac{\omega_n^{\ 2}}{s(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2})} = \frac{K_1}{s} + \frac{K_2(s + \zeta\omega_n) + K_3\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^{\ 2}(1-\zeta^2)}$$

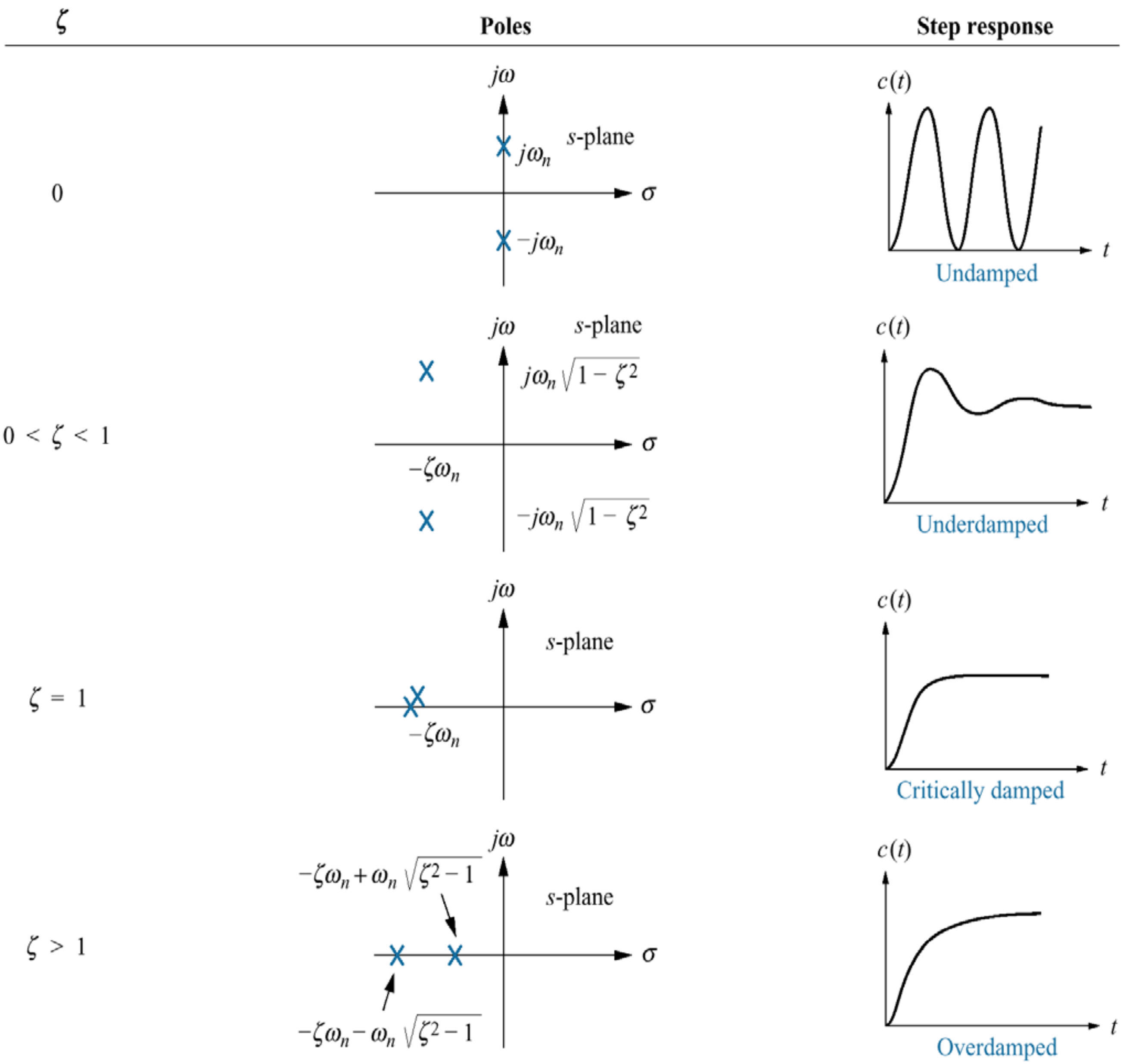
$$= \frac{1}{s} + \frac{-1\cdot(s + \zeta\omega_n) - \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^{\ 2}(1-\zeta^2)}$$

Taking the inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \phi\right)$$

where $$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$



Cases (-1 < ζ < 0) and ($\zeta \leq$ -1), when system is unstable, are not considered here.

- **Peak time**: The time required to reach the first, or maximum, peak.
- **Percent overshoot**: The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
- **Settling time**: The time required for the transient's damped oscillations to reach and stay within 2% (or 5%) of the steady-state value.
- **Rise time**: The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Evaluation of peak time:

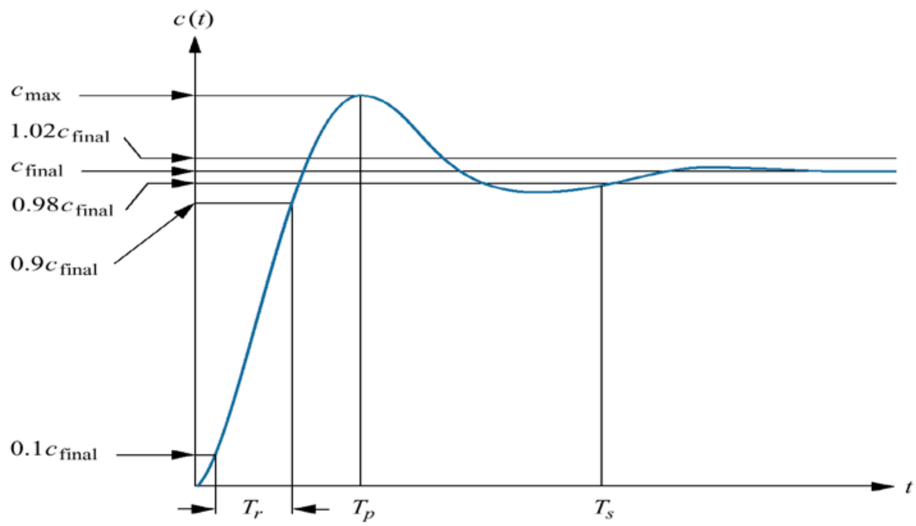
$$L[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\,t$$

Setting the derivative equal to zero yields

$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi \quad or \quad t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Peak time: $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$



Especificaciones del sistema de segundo orden

Evaluation of percent overshoot ($\%OS = \frac{c_{\max} - c_{final}}{c_{final}} \times 100$ **)**:

$$c_{\max} = c(t_p) = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \Rightarrow \quad \%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

Evaluation of settling time:
The settling time is the time it takes for the amplitude of the decaying sinusoid to reach o.o2, or

$$e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} = 0.02 \qquad \Rightarrow \quad t_s = \frac{-\ln(0.02)\sqrt{1-\zeta^2}}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

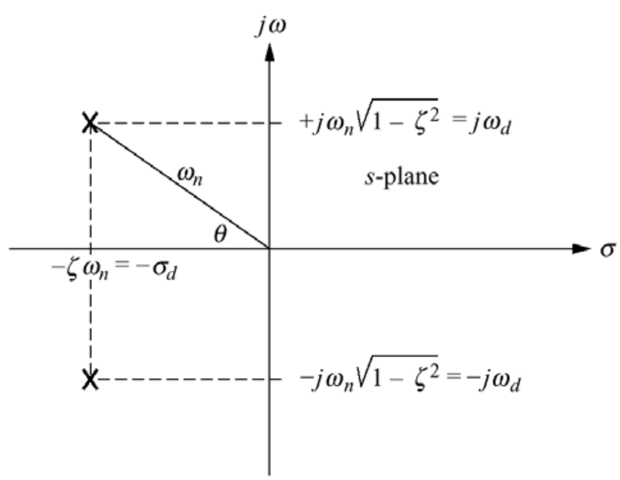
,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \qquad\qquad t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

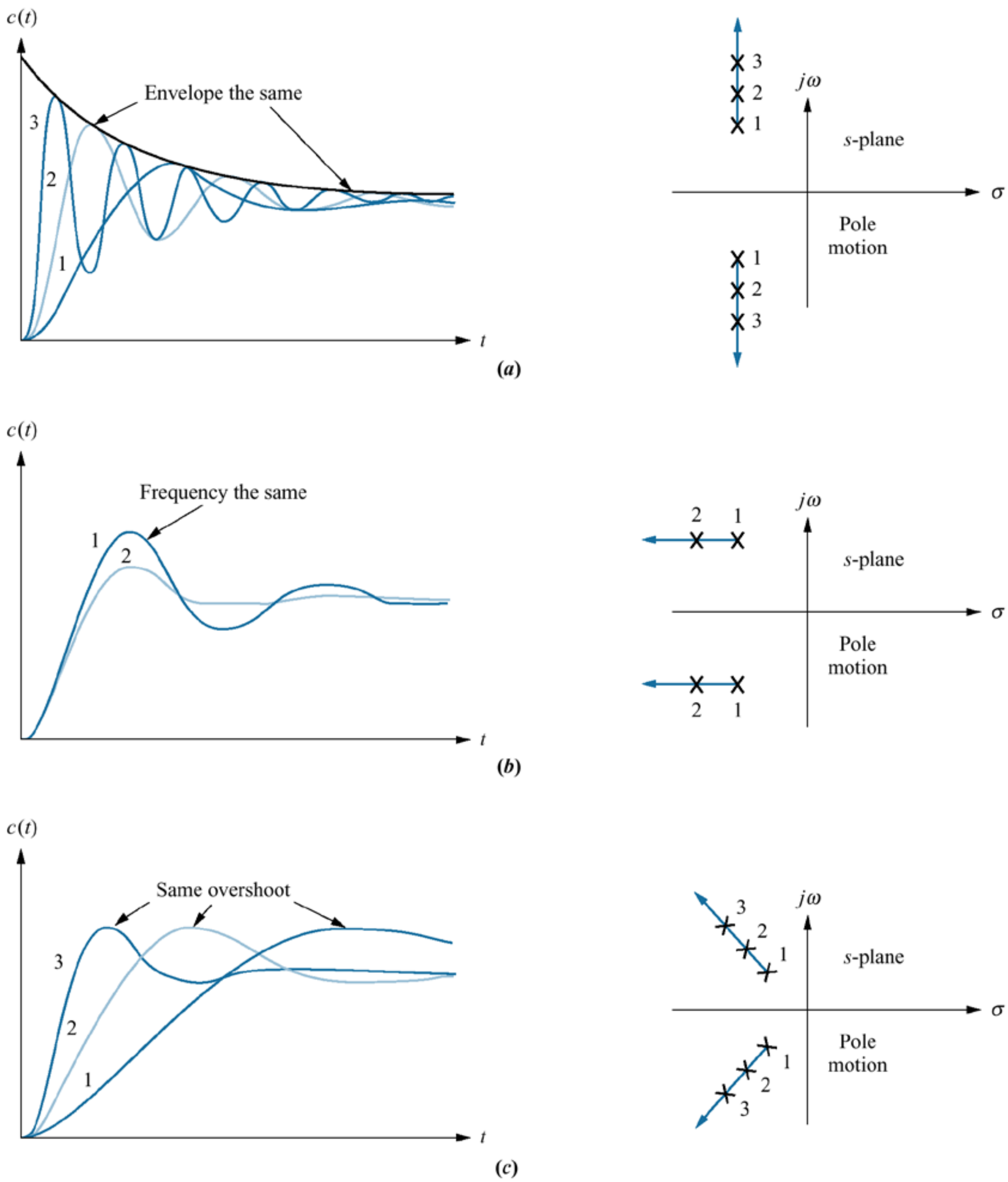
where ω_d is the imaginary part of the pole and is called the damped frequency of oscillation, and σ_d is the magnitude of the real part of the pole and is the exponential damping frequency.

Ubicación de los polos

$$G(s) = \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}} = \frac{\omega_n^{\;2}}{(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})}$$



Step responses of second-order underdamped systems as poles move:
a. with constant real part;
b. with constant imaginary part;
c. with constant damping ratio

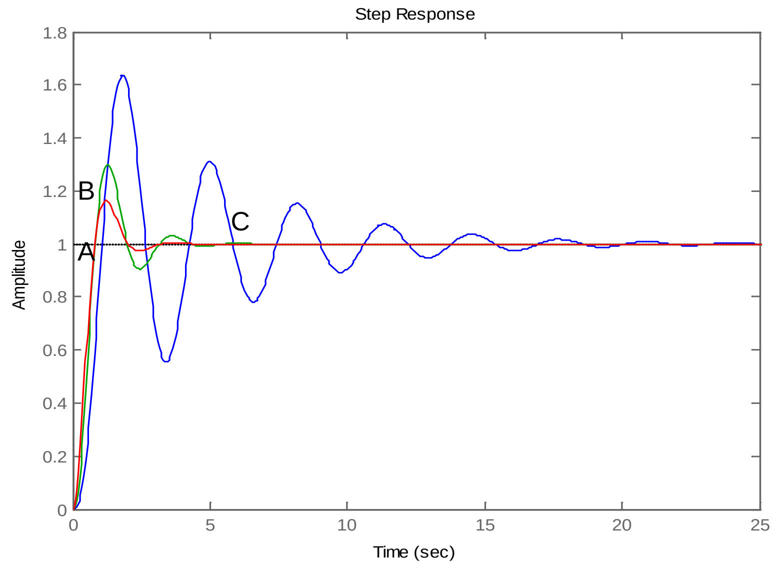


Efecto de añadir un polo en lazo abierto (simulando la F. de T. resultante en lazo cerrado)

A. Transfer function:
$$\frac{9}{s^2 + 3 s + 9}$$

B. Transfer function:
$$\frac{9}{0.1 s^3 + 1.3 s^2 + 3 s + 9}$$

C. Transfer function:
$$\frac{9}{0.5 s^3 + 2.5 s^2 + 3 s + 9}$$

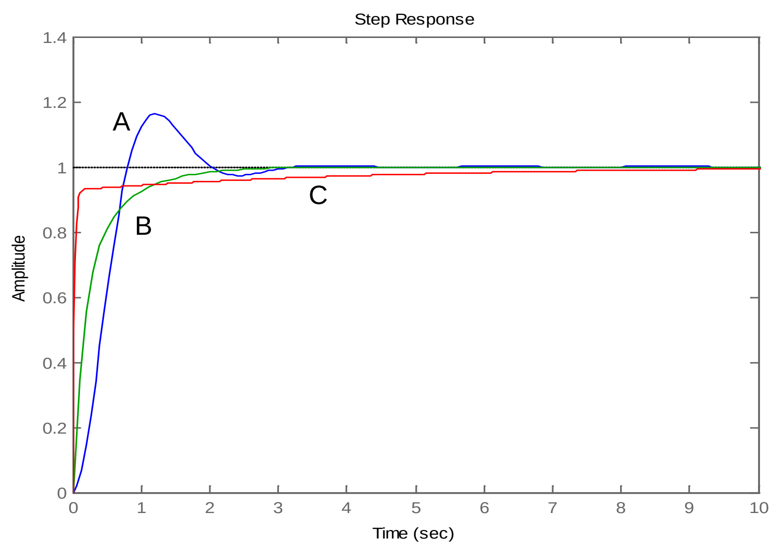


Efecto de añadir un cero en lazo abierto (simulando la F. de T. resultante en lazo cerrado)

A. Transfer function:
$$\frac{9}{s^2 + 3 s + 9}$$

B. Transfer function:
$$\frac{4.5 s + 9}{s^2 + 7.5 s + 9}$$

C. Transfer function:
$$\frac{36 s + 9}{s^2 + 39 s + 9}$$

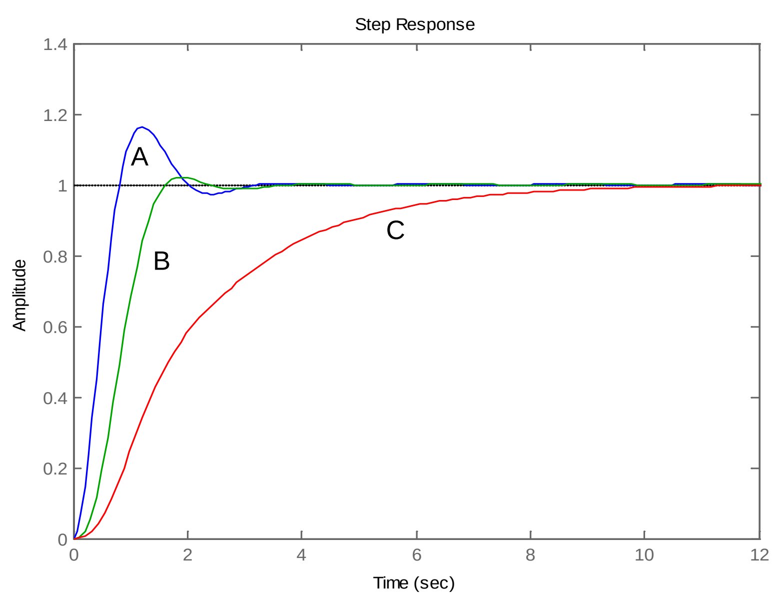


Efecto de añadir un polo en lazo cerrado (simulando la F. de T. en lazo cerrado)

A. Transfer function:
$$\frac{9}{s^2 + 3 s + 9}$$

B. Transfer function:
$$\frac{9}{0.5 s^3 + 2.5 s^2 + 7.5 s + 9}$$

C. Transfer function:
$$\frac{9}{2 s^3 + 7 s^2 + 21 s + 9}$$



Efecto de añadir un cero en lazo cerrado (simulando la F. de T. en lazo cerrado)

A. Transfer function:

$$\frac{9}{s^2 + 3 s + 9}$$

B. Transfer function:

$$\frac{2.7 s + 9}{s^2 + 3 s + 9}$$

C. Transfer function:

$$\frac{7.2 s + 9}{s^2 + 3 s + 9}$$