2. *Numerical Analysis* by S. S. Kunz. McGraw-Hill, 1964. Chapter 13 offers a clear, precise summary of some numerical methods in PDE theory.

3. *Numerical Solution of Partial Differential Equations* by G. D. Smith. Oxford University Press, 1965. A concise book describing finite difference methods in PDE theory; clearly written.

LESSON 38

An Explicit Finite-Difference Method

PURPOSE OF LESSON: To introduce the idea of explicit finite-difference methods and show how they can be used to solve hyperbolic and parabolic problems. The basic idea is that after a PDE like

$$u_t = u_{xx}$$

is replaced by its finite-difference approximation, we can solve for the solution explicitly at *one value of time* in terms of the solution at *earlier values of time*. In this way, an initial-boundary-value problem (hyperbolic or parabolic) can be solved by consecutively finding the solution at larger and larger values of time.

A problem we face is that as we make the grid sizes *small* so that the finite differences accurately represent the derivatives, the number of computations *increases*, and so the roundoff error increases.

In the previous lesson, we solved elliptic boundary-value problems (steady-state problems) where the PDE was satisfied in a given region of space, and the solution (or its derivative) was specified on the boundary. In those types of problems, we found the approximate solution at the *interior grid points* by solving a system of algebraic equations. In other words, the solution at all the interior grid points was found *simultaneously*.

In this lesson, we will show how *time-dependent problems* can be solved by finite-difference approximations. The idea here is that if we are given the solution when time is *zero*, we can then find the solution for $t = \Delta t, 2\Delta t, 3\Delta t, \ldots$ by means of a *marching process*. Replacing both the *space* and *time* derivatives by their finite-difference approximations, we can then solve for the solution $u_{i,j}$ in the difference equation *explicitly* in terms of the solution at earlier values of time. This process is called an **explicit-type marching process**, since we find the solution at a *single* value of time in terms of the solution at earlier values of time.

To show how this method works, we consider a representative problem from heat flow.

The Explicit Method for Parabolic Equations

Consider the problem of heat flow along a rod initially at temperature zero, where the left end of the rod is fixed at temperature one, and the right-hand side experiences a heat loss (or gain) proportional to the difference between the temperature at that end and an outside temperature that is given by $g(t)$. In other words, we solve the problem

$$\text{PDE} \qquad u_t = u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 1 \\ u_x(1,t) = -[u(1,t) - g(t)] \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 0 \qquad 0 \le x \le 1$$

To solve this problem by finite differences, we start by drawing the usual rectangular grid system with grid points:

$$x_j = jh \qquad j = 0, 1, 2, \ldots, n$$
$$t_i = ik \qquad i = 0, 1, 2, \ldots, m$$

See Figure 38.1.

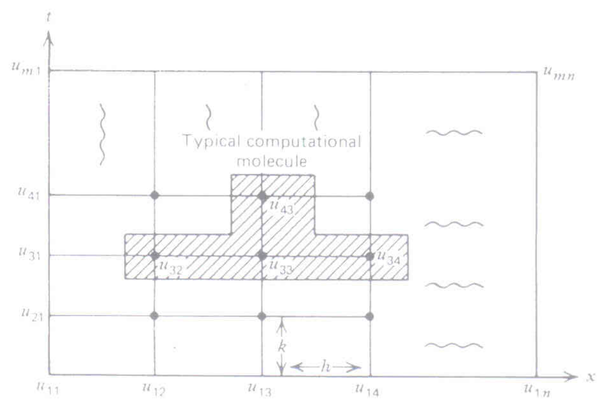

FIGURE 38.1 Grid system for a heat-flow problem.

Note that in Figure 38.1, the $u_{i,j}$ on the *left* and *bottom* are given BCs and ICs, and our job is to find the other $u_{i,j}$'s. To do this, we begin by replacing the partial derivatives u_t and u_{xx} in the heat equation with their approximations

$$u_t = \frac{1}{k}[u(x,t+k) - u(x,t)] = \frac{1}{k}(u_{i+1,j} - u_{i,j})$$

$$u_{xx} = \frac{1}{h^2}[u(x+h,t) - 2u(x,t) + u(x-h,t)] = \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})$$

By substituting these expressions into $u_t = u_{xx}$ and solving for the solution at the largest value of time, we have

$$(38.2) \qquad \boxed{u_{i+1,j} = u_{i,j} + \frac{k}{h^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]}$$

This is the formula we are looking for, since it gives us the solution at one value of time in terms of the solution at earlier values of time (note that the index i stands for time). Figure 38.1 shows those values of the solution that are involved in the formula.

We are now almost ready to begin the computations. First, however, we must approximate the derivative in the right-hand BC

$$u_x(1,t) = -[u(1,t) - g(t)]$$

by

$$(38.3) \qquad \frac{1}{h}[u_{i,n} - u_{i,n-1}] = -[u_{i,n} - g_i]$$

where $g_i = g(ik)$ is given. Note that we have replaced $u_x(1,t)$ by the *backward-difference approximation*, since the forward-difference approximation would require knowing values of $u_{i,j}$ outside the domain. Solving now for $u_{i,n}$ in this BC gives us

$$(38.4) \qquad u_{i,n} = \frac{u_{i,n-1} + hg_i}{1 + h}$$

With this equation and our explicit formula (38.2), we are ready to begin the computations.

Algorithm for the Explicit Method

STEP 1 Find the solution at the grid points for $t = \Delta t$ by using the explicit formula

$$u_{2,j} = u_{1,j} + \frac{k}{h^2}[u_{1,j+1} - 2u_{1,j} + u_{1,j-1}] \qquad j = 2, 3, \ldots, n-1$$

See Figure 38.2.

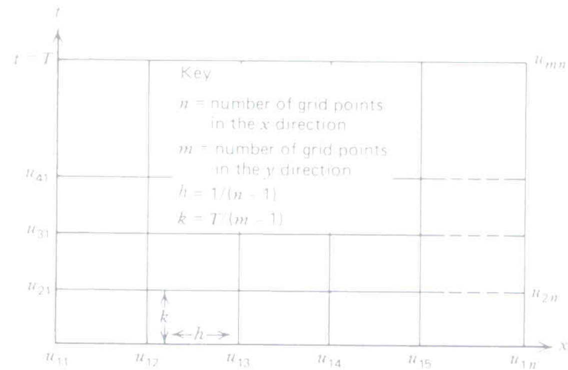

FIGURE 38.2 Diagram illustrating the explicit method.

STEP 2 Find $u_{2,n}$ from formula (38.4)

$$u_{2,n} = \frac{u_{2,n-1} + hg_2}{1 + h}$$

Steps 1 and 2 find the solution for $t = \Delta t$. To find the solution for $t = 2\Delta t$ (second row from the bottom in Figure 38.2), repeat steps 1 and 2, moving up one more row (increase i by 1) and using the values of $u_{i,j}$ just computed; for $t = 3\Delta t, 4\Delta t, \ldots$, keep repeating the same process.

In order for the reader to be able to computerize this method, we will present a fairly detailed flow diagram of the method in Figure 38.3. Those students not familiar with flow diagrams should think of them as links between computational algorithms and detailed computer programs. Flow diagrams explain in a precise manner how the computations should be carried out.

NOTES

1. There is a serious deficiency in the explicit method, for if the step size in t is large compared to the step size in x, then machine roundoff error can grow until it ruins the accuracy of the solution. The relative size of these two numbers x and t depends on the particular equation and the BCs, but, generally, the step size in t should be much smaller than the step size in x. In reference 3 of the recommended reading in Lesson 37, the author proves that we must have $k/h^2 \leq 0.5$ in order for this method to work.
2. A general rule of thumb is that as the step sizes Δt and Δx are made smaller, the *truncation error* of approximating partial derivatives by finite differences decreases. However, the smaller these grid sizes, the more computations necessary, and, hence, the *roundoff error*, as a result of rounding off our computations, will increase. Therefore, we have the phenomenon illustrated in Figure 38.5.

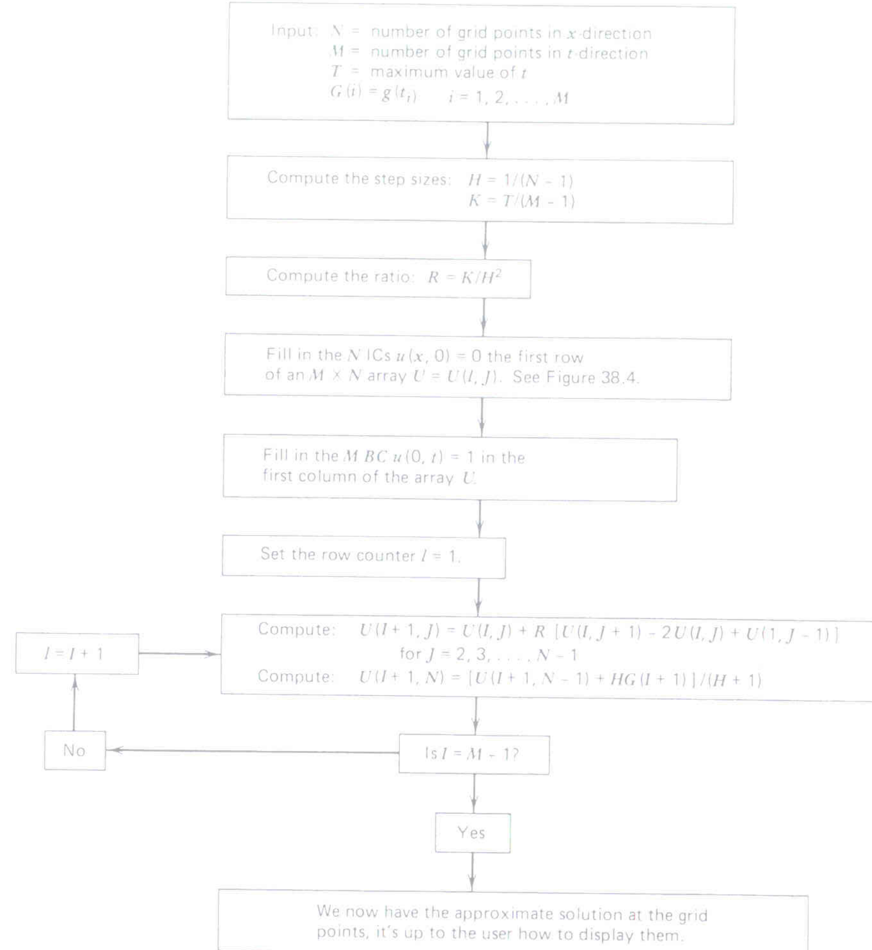

FIGURE 38.3 Flow diagram of the explicit method.

3. The hyperbolic problem

PDE	$u_{tt} = u_{xx} \quad 0 < x < 1$	$0 < t < \infty$
BCs	$\begin{cases} u(0,t) = g_1(t) \\ u(1,t) = g_2(t) \end{cases}$	$0 < t < \infty$
ICs	$\begin{cases} u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$	$0 \leq x \leq 1$

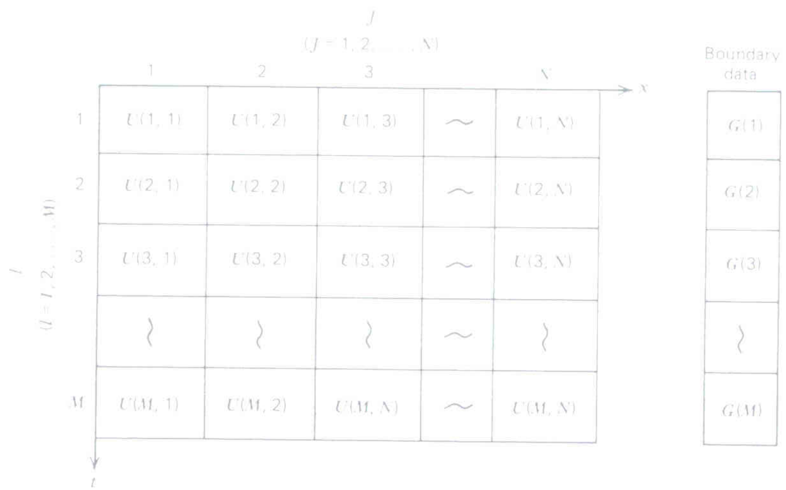

FIGURE 38.4 Arrays used in the explicit method.

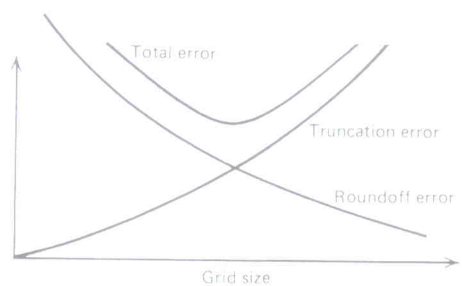

FIGURE 38.5 Total error as a function of grid size.

can also be solved by the explicit finite-difference method. Here, we can approximate the derivatives u_{tt} and u_{xx} by

$$u_{tt} \cong \frac{1}{k^2}[u(x,t+k) - 2u(x,t) + u(x,t-k)]$$

$$u_{xx} \cong \frac{1}{h^2}[u(x+h,t) - 2u(x,t) + u(x-h,t)]$$

and the derivative $u_t(x,0)$ in the IC by

$$u_t(x,0) \cong \frac{1}{k}[u(x,k) - u(x,0)] = \frac{1}{k}[u(x,k) - \phi(x)]$$

Hence, solving for $u(x,t+k)$ explicitly in terms of the solution at earlier values of time gives

$$u(x,t+k) = 2u(x,t) - u(x,t-k) + \left(\frac{k}{h}\right)^2 [u(x+h,t) - 2u(x,t) + u(x-h,t)] \tag{38.5}$$

From this equation, it is clear that we must already know the solution at *two* previous time steps, and, hence, we must use the initial-velocity condition

$$\frac{1}{k}[u(x,k) - \phi(x)] = \psi(x)$$

to get us started. Solving for $u(x,k)$ gives $u(x,k) = \phi(x) + k\psi(x)$, and, thus, we can find the solution for $t = \Delta t$. The solution at all later values of time can now be found by our explicit formula (38.5).

PROBLEMS

1. Find the finite-difference solution of the heat-conduction problem

$$\text{PDE} \quad u_t = u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = \sin(\pi x) \quad 0 \le x \le 1$$

for $t = 0.005, 0.010, 0.015$ by the explicit method. Let $h = \Delta x = 0.1$. Plot the solution at $x = 0, 0.1, 0.2, 0.3, \ldots, 0.9, 1$ for $t = 0.015$.

2. Solve problem 1 analytically (separation of variables) and evaluate the analytical solution at the grid points: $x = 0, 0.1, 0.2, \ldots, 0.9, 1$ for $t = 0.015$. Compare these results to your numerical solution in problem 1. (You may wish to write a small computer program or use a calculator to evaluate the separation-of-variables solution.)
3. Write a flow diagram to carry out the computations of the hyperbolic problem discussed in note 3 of this lesson.
4. Do problem 1 except now *replace* the BC at $x = 1$ by

$$u_x(1,t) = -[u(1,t) - 1]$$