OTHER READING

Finite-Difference Methods for PDEs by G. F. Forsythe and W. R. Wasow. John Wiley & Sons, 1960. An excellent text with several physical examples illustrated, soil-drainage problems, oil-flow problems, and a meteorological-forecast problem are a few of the problems discussed.

LESSON 39

An Implicit Finite-Difference Method (Crank-Nicolson Method)

PURPOSE OF LESSON: To show how time-dependent problems can be solved by another finite-difference scheme known as *implicit methods*. In this method, we again replace the partial derivatives in the problem by their finite-difference approximations, but unlike explicit methods (where we solved for $u_{i+1,j}$ explicitly in terms of earlier values), in implicit methods, we solve *a system of equations* in order to find the solution at the largest value of time. In other words, for each new value of time we solve a system of algebraic equations to find *all* the values.

Implicit methods have an advantage over explicit ones, since the step size can be made larger without worrying about excessive buildup of round-off error.

A popular implicit method known as the *Crank-Nicholson method* will be used to solve a parabolic problem.

The difficulty with the explicit methods that we discussed in the last lesson is that the step size in time must be small in order for the method to work properly. In particular, if we were to solve the simple heat-flow problem

$$\text{PDE} \quad u_t = u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{(39.1)} \qquad \text{BC} \quad \begin{cases} u(0,t) = g_1(t) \\ u(1,t) = g_2(t) \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = f(x) \qquad 0 \leqslant x \leqslant 1$$

by the *explicit method*, it would be necessary for the grid sizes Δt and Δx to satisfy

$$\frac{\Delta t}{(\Delta x)^2} \leqslant 0.5$$

in order for the method to be numerically stable (the roundoff errors don't build up). See reference 1 (p. 45) of the recommended reading for details of numerical stability. In other words, if the grid size Δx in the x-direction were chosen to be $\Delta x = 0.1$, then the time increment Δt must be $\Delta t \le 0.5\Delta x^2 = 0.005$ (hence, to go from $t = 0$ to $t = 1$ would take 200 steps).

There are, however, procedures (implicit methods) that allow us to take larger steps by doing more work per step; in these methods, we can take relatively large steps by solving a *system of algebraic equations* at each step. To illustrate how these methods work, we solve the following heat-flow problem.

The Heat-Flow Problem Solved by an Implicit Method

Consider the following problem:

$$
\text{(39.2)}\quad
\begin{array}{lll}
\text{PDE} & u_t = u_{xx} & 0 < x < 1 \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = 1 & 0 \le x \le 1
\end{array}
$$

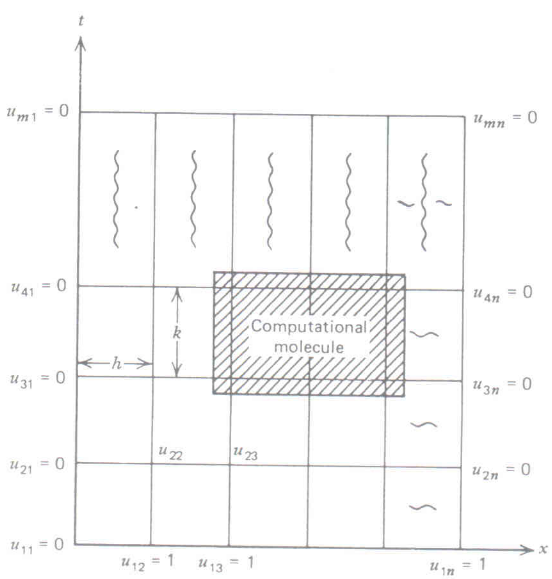

FIGURE 39.1 Grid system for implicit scheme ($\Delta x = 0.2$).

We replace the partial derivatives u_t and u_{xx} by the following approximations:

$$u_t(x,t) = \frac{1}{k}[u(x,t+k) - u(x,t)]$$

$$u_{xx}(x,t) = \frac{\lambda}{h^2}[u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)] + \frac{(1-\lambda)}{h^2}[u(x+h,t) - 2u(x,t) + u(x-h,t)]$$

where λ is a chosen number in the interval [0,1]. Note that our approximation for u_{xx} is a *weighted average* of the central-difference approximation to the derivative u_{xx} at time values t and $t + k$. In the special case when $\lambda = 0.5$, it is just the ordinary average of these two central differences, while if $\lambda = 0.75$, our approximation puts weights of 0.75 and 0.25 on each of the two terms (note, if $\lambda = 0$, it is the usual *explicit* finite-difference method we used in the last lesson).

If we now substitute the approximations for u_t and u_{xx} into our problem, we get the new *finite-difference problem*

$$
\text{(39.3)}\quad
\begin{array}{ll}
\text{Difference equation} & \frac{1}{k}(u_{i+1,j} - u_{i,j}) \\
& = \frac{\lambda}{h^2}(u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}) + \frac{(1-\lambda)}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \\
\text{BC} & \begin{cases} u_{i,1} = 0 \\ u_{i,n} = 0 \end{cases} \quad i = 1, 2, \ldots, m \\
\text{IC} & u_{1,j} = 1 \quad j = 2, \ldots, n-1
\end{array}
$$

See Figure 39.1.

Now, if we rewrite the difference equation in (39.3), putting the $u_{i,j}$'s with the largest time subscript (i-subscript) on the left-hand side of the equation, we arrive at the equation

$$
\text{(39.4)}\quad -\lambda r u_{i+1,j+1} + (1 + 2r\lambda)u_{i+1,j} - \lambda r u_{i+1,j-1} = r(1-\lambda)u_{i,j+1} + [1 - 2r(1-\lambda)]u_{i,j} + r(1-\lambda)u_{i,j-1}
$$

where we have set $r = k/h^2$ for convenience. Note that for a *fixed subscript* i and for j going from 2 to $n - 1$, this is a system of $n - 2$ equations in the $n - 2$ unknowns $u_{i+1,2}, u_{i+1,3}, u_{i+1,4}, \ldots u_{i+1,n-1}$ [which are the interior grid points at $t = (i + 1)\Delta t$].

To help show exactly what $u_{i,j}$'s are involved in this formula, we write it in the symbolic or molecular form shown in Figure 39.2.

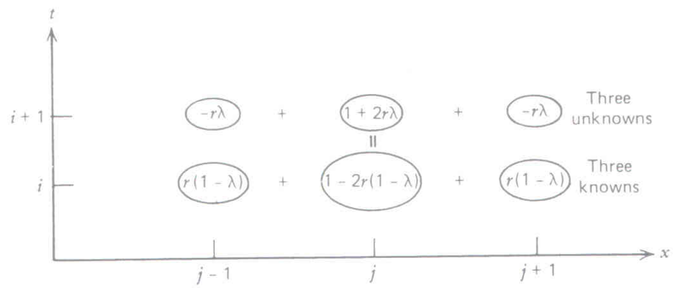

FIGURE 39.2 The molecular form of the implicit formula.

We now show how equation (39.4) can be used to find the solution of problem (39.2).

Implicit Algorithm for Heat Problem (39.2)

STEP 1 Pick some value for λ ($0 \leq \lambda \leq 1$). Note that if $\lambda = 0$, then equation (39.4) is the same as the explicit formula we developed in lesson 38.

STEP 2 Pick $h = \Delta x = 0.2$ and $k = \Delta t = 0.08$ ($r = k/h^2 = 2$). This gives *six* grid points in the x-direction (four *interior* grid points); see Figure 39.1. Also let's pick the weight parameter $\lambda = 0.5$ (which is called the Crank-Nicolson method). If we now apply our computational molecule to the first and second rows ($i = 1$), moving it from left to right ($j = 2, 3, 4, 5$), we get the following four equations:

$$\begin{aligned}
-u_{21} + 3u_{22} - u_{23} &= u_{11} - u_{12} + u_{13} = 1 \\
-u_{22} + 3u_{23} - u_{24} &= u_{12} - u_{13} + u_{14} = 1 \\
-u_{23} + 3u_{24} - u_{25} &= u_{13} - u_{14} + u_{15} = 1 \\
-u_{24} + 3u_{25} - u_{26} &= u_{14} - u_{15} + u_{16} = 1
\end{aligned}$$

which if written in matrix form, placing the four unknown interior grid points u_{22}, u_{23}, u_{24}, and u_{25} on the left-hand side of the equation, gives

$$(39.5) \qquad \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_{22} \\ u_{23} \\ u_{24} \\ u_{25} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This system of equations is called a **tridiagonal** system, and to solve it, we use a method that transforms a tridiagonal system of the form

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_2 & \mathbf{c_2} & 0 & \cdots & 0 \\ 0 & a_2 & \mathbf{b_3} & \mathbf{c_3} & 0 & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & c_{n-1} \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix}$$

into an equivalent one

$$\begin{bmatrix} 1 & c_1^* & 0 & 0 & \cdots & 0 \\ 0 & 1 & c_2^* & 0 & \cdots & 0 \\ 0 & 0 & 1 & c_3^* & & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & c_{n-1}^* \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} d_1^* \\ d_2^* \\ d_3^* \\ \cdot \\ \cdot \\ \cdot \\ d_n^* \end{bmatrix}$$

where

$$c_1^* = c_1/b_1 \qquad c_{j+1}^* = \frac{c_{j+1}}{b_{j+1} - a_j c_j^*} \qquad j = 1, 2, \ldots, n-2$$

and

$$d_1^* = d_1/b_1 \qquad d_{j+1}^* = \frac{d_{j+1} - a_j d_j^*}{b_{j+1} - a_j c_j^*} \qquad j = 1, 2, \ldots, n-1$$

There is nothing magical about this transformation; it just involves rewriting the original system of equations in an equivalent form. The point is, once we have written the system of equations in the new form, it is easy to solve. Solving from bottom to top, we have

$$x_n = d_n^* \qquad x_j = d_j^* - c_j^* x_{j+1} \qquad j = n-1, \quad n-2, \ldots, 2, 1$$

Applying this method to our system of four equations (39.5), we get:

$$\begin{aligned}
u_{22} &= 0.60 \\
u_{23} &= 0.80 \\
u_{24} &= 0.80 \\
u_{25} &= 0.60
\end{aligned}$$

This gives us the solution (approximation) at the interior grid points for $t = \Delta t$. After finding these values, we move to the next time value and solve a new set of equations.

This implicit method takes more work at each value of time than does the explicit method, but it enables us to pick a larger Δt and still get a good approximation.

PROBLEMS

1. Derive equation (39.4) from the difference equation in (39.3).
2. Tell how you would solve the problem

$$\text{PDE} \qquad u_t = u_{xx} \qquad 0 < x < 1$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 1 \\ u_x(1,t) + u(1,t) = g(t) \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 0 \qquad 0 \leq x \leq 1$$

by the implicit finite-difference method.

3. How would you solve

$$\text{PDE} \qquad u_t = u_{xx} + u \qquad 0 < x < 1$$

$$\text{BCs} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = 1 \qquad 0 \leq x \leq 1$$

by the implicit method?

4. What is the molecular form of equation (39.4) when we pick $\lambda = 1$?
5. Write a flow diagram to solve heat-flow problem (39.2). Write a computer program if facilities are available. A good experiment would be to solve this problem numerically with a simple IC $u(x,0) = \sin(\pi x)$ for different values of the parameter λ. You could compare the true analytical solution, which, in this case, is

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

with the numerical solution for different values of λ.

6. Solve the system of algebraic equations (39.5) using the formulas given in the lesson.

OTHER READING

Numerical Methods in PDEs by W. F. Ames. Academic Press, 1977. An excellent book with applications to fluid dynamics and elasticity.