LESSON 1

Introduction to Partial Differential Equations

> **PURPOSE OF LESSON:** To show what partial differential equations are, why they are useful, and how they are solved; also included is a brief discussion on how they are classified as various kinds and types. An overview is given of many of the ideas that will be studied in detail later.

Most physical phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics, or heat flow, can be described in general by partial differential equations (PDEs); in fact, most of mathematical physics are PDEs. It's true that simplifications can be made that reduce the equations in question to ordinary differential equations, but, nevertheless, the complete description of these systems resides in the general area of PDEs.

What Are PDEs?

A partial differential equation is an equation that contains partial derivatives. In contrast to ordinary differential equations (ODEs), where the unknown function depends only on *one variable,* in PDEs, the unknown function depends on several variables (like temperature $u(x,t)$ depends both on location x and time t).

Let's list some well-known PDEs; note that for notational simplicity we have called

$$u_t = \frac{\partial u}{\partial t} \qquad u_x = \frac{\partial u}{\partial x} \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad \cdots$$

A Few Well-Known PDEs

$u_t = u_{xx}$ (heat equation in one dimension)

$u_t = u_{xx} + u_{yy}$ (heat equation in two dimensions)

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (Laplace's equation in polar coordinates)

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \quad \text{(wave equation in three dimensions)}$$

$$u_{tt} = u_{xx} + \alpha u_t + \beta u \quad \text{(telegraph equation)}$$

Note on the Examples

The unknown function u always depends on *more* than one variable. The variable u (which we differentiate) is called the **dependent** variable, whereas the ones we differentiate *with respect to* are called the **independent** variables. For example, it is clear from the equation

$$u_t = u_{xx}$$

that the dependent variable $u(x,t)$ is a function of two independent variables x and t, whereas in the equation

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$u(r,\theta,t)$ depends on r, θ, and t.

Why Are PDEs Useful?

Most of the **natural laws of physics**, such as Maxwell's equations, Newton's law of cooling, the Navier-Stokes equations, Newton's equations of motion, and Schrodinger's equation of quantum mechanics, are stated (or can be) in terms of PDEs, that is, these laws describe physical phenomena by relating **space and time derivatives.** Derivatives occur in these equations because the derivatives represent *natural things* (like velocity, acceleration, force, friction, flux, current). Hence, we have equations relating partial derivatives of some unknown quantity that we would like to find.

The purpose of this book is to show the reader two things

1. How to *formulate* the PDE from the physical problem (constructing the mathematical model).
2. How to *solve* the PDE (along with initial and boundary conditions).

We wait a few lessons before we start the modeling problem; now, a brief overview on how PDEs are solved.

How Do You Solve a Partial Differential Equation?

This is a good question. It turns out that there is an entire arsenal of methods available to the practitioner; the most important methods are those that change PDEs into ODEs. Ten useful techniques are

1. *Separation of Variables.* This technique reduces a PDE in n variables to n ODEs.
2. *Integral Transforms.* This procedure reduces a PDE in n independent variables to one in $n - 1$ variables; hence, a PDE in two variables could be changed to an ODE.
3. *Change of Coordinates.* This method changes the original PDE to an ODE or else another PDE (an easier one) by changing the coordinates of the problem (rotating the axis and things like that).
4. *Transformation of the Dependent Variable.* This method transforms the unknown of a PDE into a new unknown that is easier to find.
5. *Numerical Methods.* These methods change a PDE to a system of *difference equations* that can be solved by means of iterative techniques on a computer; in many cases, this is the only technique that will work. In addition to methods that replace PDEs by difference equations, there are other methods that attempt to approximate solutions by polynomial surfaces (spline approximations).
6. *Pertubation Methods.* This method changes a nonlinear problem into a sequence of *linear ones* that approximates the nonlinear one.
7. *Impulse-response Technique.* This procedure decomposes initial and boundary conditions of the problem into *simple impulses* and finds the response to each impulse. The overall response is then found by adding these simple responses.
8. *Integral Equations.* This technique changes a PDE to an **integral equation** (an equation where the unknown is inside the integral). The integral equation is then solved by various techniques.
9. *Calculus of Variations Methods.* These methods find the solution to PDEs by reformulating the equation as a *minimization problem.* It turns out that the minimum of a certain expression (very likely the expression will stand for total energy) is also the solution to the PDE.
10. *Eigenfunction Expansion.* This method attempts to find the solution of a PDE as an infinite sum of *eigenfunctions*. These eigenfunctions are found by solving what is known as an eigenvalue problem corresponding to the original problem.

Kinds of PDEs

Partial differential equations are classified according to many things. Classification is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Six basic classifications are

1. *Order of the PDE.* The order of a PDE is the order of the *highest partial derivative* in the equation, for example,

$$u_t = u_{xx} \quad \text{(second order)}$$

$$u_t = u_x \quad \text{(first order)}$$
$$u_t = uu_{xxx} + \sin x \quad \text{(third order)}$$

2. *Number of Variables.* The number of variables is the *number of independent* variables, for example,

$$u_t = u_{xx} \quad \text{(two variables: } x \text{ and } t)$$
$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \quad \text{(three variables: } r,\ \theta, \text{ and } t)$$

3. *Linearity.* Partial differential equations are either *linear* or *nonlinear*. In the linear ones, the dependent variable u and all its derivatives appear in a linear fashion (they are not multiplied together or squared, for example). More precisely, a **second-order linear equation in two variables** is an equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.1}$$

where A, B, C, D, E, F, and G can be *constants* or given *functions* of x and y; for example,

$$u_{tt} = e^{-t}u_{xx} + \sin t \quad \text{(linear)}$$
$$uu_{xx} + u_t = 0 \quad \text{(nonlinear)}$$
$$u_{xx} + yu_{yy} = 0 \quad \text{(linear)}$$
$$xu_x + yu_y + u^2 = 0 \quad \text{(nonlinear)}$$

4. *Homogeneity.* The equation (1.1) is called **homogeneous** if the right-hand side $G(x,y)$ is identically zero for all x and y. If $G(x,y)$ is not identically zero, then the equation is called **nonhomogeneous**.
5. *Kinds of Coefficients.* If the coefficients A, B, C, D, E, and F in equation (1.1) are constants, then (1.1) is said to have **constant coefficients** (otherwise, variable coefficients).
6. *Three Basic Types of Linear Equations.* All linear PDEs like equation (1.1) are either
   (a) parabolic
   (b) hyperbolic
   (c) elliptic

Parabolic. **Parabolic equations** describe heat flow and diffusion processes and satisfy the property $B^2 - 4AC = 0$.

Hyperbolic. **Hyperbolic equations** describe vibrating systems and wave motion and satisfy the property $B^2 - 4AC > 0$.

Elliptic. **Elliptic equations** describe *steady-state* phenomena and satisfy the property $B^2 - 4AC < 0$.

Examples.

(a) $u_t = u_{xx}$ $\quad B^2 - 4AC = 0$ (parabolic)

(b) $u_{tt} = u_{xx}$ $\quad B^2 - 4AC = 4$ (hyperbolic)

(c) $u_{\xi\eta} = 0$ $\quad B^2 - 4AC = 1$ (hyperbolic)

(d) $u_{xx} + u_{yy} = 0$ $\quad B^2 - 4AC = -4$ (elliptic)

(e) $yu_{xx} + u_{yy} = 0$ $\quad B^2 - 4AC = -4y$ $\quad \begin{cases} \text{elliptic for } y > 0 \\ \text{parabolic for } y = 0 \\ \text{hyperbolic for } y < 0 \end{cases}$

(In the case of variable coefficients, the situation can change from point to point.)

NOTES

1. In general, $B^2 - 4AC$ is a *function* of the independent variables; hence, an equation can change from one basic type to another throughout the domain of the equation (although it's not common).
2. The general linear equation (1.1) was written with independent variables x and y. In many problems, one of the two variables stands for time and hence would be written in terms of x and t.
3. A *general classification* diagram is given in Figure 1.1.

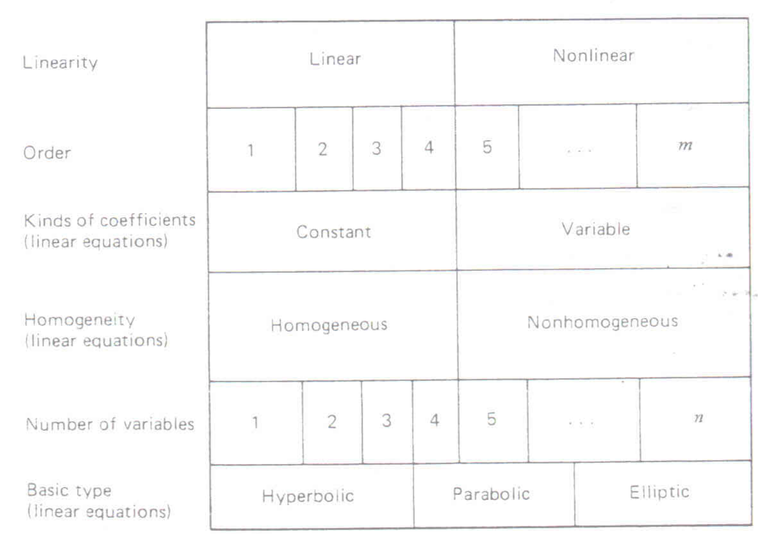

FIGURE 1.1 Classification diagram for partial differential equations.

PROBLEMS

1. Classify the following equations according to all the properties we've discussed in Figure 1.1:

   (a) $u_t = u_{xx} + 2u_x + u$
   (b) $u_t = u_{xx} + e^{-t}$
   (c) $u_{xx} + 3u_{xy} + u_{yy} = \sin x$
   (d) $u_{tt} = uu_{xxxx} + e^{-t}$

2. How many solutions to the PDE $u_t = u_{xx}$ can you find? Try solutions of the form $u(x,t) = e^{ax + bt}$.
3. If $u_1(x,y)$ and $u_2(x,y)$ satisfy equation (1.1), then is it true that the sum satisfies it?; if yes, prove it.
4. Probably the easiest of all PDEs to solve is the equation

$$\frac{\partial u(x,y)}{\partial x} = 0$$

   Can you solve this equation? (Find all functions $u(x,y)$ that satisfy it.)
5. What about the PDE

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0$$

   Can you find all solutions $u(x,y)$ to this equation? (How many are there?) How does this compare with an ODE like

$$\frac{d^2 y}{dx^2} = 0$$

   insofar as the number of solutions is concerned?

OTHER READING

1. *Elementary Partial Differential Equations* by P. W. Berg and J. L. McGregor. Holden-Day, 1966. Clearly written with several nice problems; a nice book to own.
2. *Analysis and Solution of Partial Differential Equations* by R. L. Street. Brooks-Cole, 1973. A well-written text covering many of the topics we will cover in this book.

PART 2

Diffusion-Type Problems