4

The Finite Volume Method for Diffusion Problems

4.1 Introduction

The nature of the transport equations governing fluid flow and heat transfer and the formal control volume integration were described in Chapter 2. Here we develop the numerical method based on this integration, the **finite volume (or control volume) method**, by considering the simplest transport process of all: pure diffusion in the steady state. The governing equation of steady diffusion can easily be derived from the general transport equation (2.39) for property ϕ by deleting the transient and convective terms. This gives

$$div(\Gamma\ grad\ \phi) + S_\phi = 0 \tag{4.1}$$

The control volume integration, which forms the key step of the finite volume method that distinguishes it from all other CFD techniques, yields the following form:

$$\int_{CV} div(\Gamma\ grad\ \phi)dV + \int_{CV} S_\phi dV = \int_A \mathbf{n}\ .\ (\Gamma\ grad\ \phi)dA + \int_{CV} S_\phi dV = 0 \tag{4.2}$$

By working with the one-dimensional steady state diffusion equation the approximation techniques that are needed to obtain the so-called discretised equations are introduced. Later the method is extended to two- and three-dimensional diffusion problems. Application of the method to simple one-dimensional steady state heat transfer problems is illustrated through a series of worked examples and the accuracy of the method is gauged by comparing numerical results with analytical solutions.

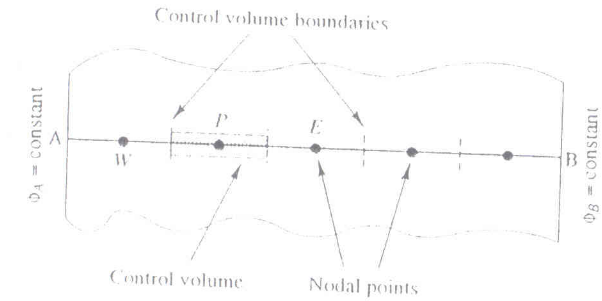
4.2 Finite volume method for one-dimensional steady state diffusion

Consider the steady state diffusion of a property ϕ in a one-dimensional domain defined in Figure 4.1. The process is governed by

$$\frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + S = 0 \tag{4.3}$$

where Γ is the diffusion coefficient and S is the source term. Boundary values of ϕ at points A and B are prescribed. An example of this type of process, one-dimensional heat conduction in a rod, is studied in detail in section 4.3.

Fig. 4.1



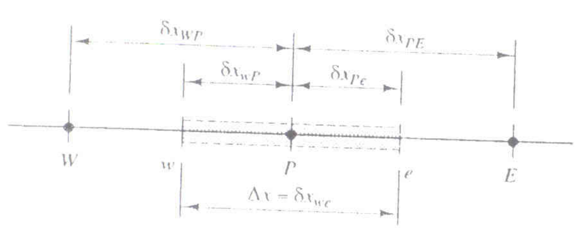
Step 1: Grid generation

The first step in the finite volume method is to divide the domain into discrete control volumes. Let us place a number of nodal points in the space between A and B. The boundaries (or faces) of control volumes are positioned mid-way between adjacent nodes. Thus each node is surrounded by a control volume or cell. It is common practice to set up control volumes near the edge of the domain in such a way that the physical boundaries coincide with the control volume boundaries.

At this point it is appropriate to establish a system of notation that can be used in future developments. The usual convention of CFD methods is shown in Figure 4.2.

A general nodal point is identified by P and its neighbours in a one-dimensional geometry, the nodes to the west and east, are identified by W and E respectively. The west side face of the control volume is referred to by 'w' and the east side control volume face by 'e'. The distances between the nodes W and P, and between nodes P and E, are identified by δx_{WP} and δx_{PE} respectively. Similarly the distances between face w and point P and between P and face e are denoted by δx_{wP} and δx_{Pe} respectively. Figure 4.2 shows that the control volume width is $\Delta x = \delta x_{we}$.

Fig. 4.2



Step 2: discretisation

The key step of the finite volume method is the integration of the governing equation (or equations) over a control volume to yield a discretised equation at its nodal point P. For the control volume defined above this gives

$$\int_{\Delta V} \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) dV + \int_{\Delta V} S\,dV = \left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w + \bar{S}\Delta V = 0 \tag{4.4}$$

Here A is the cross-sectional area of the control volume face, ΔV is the volume and $\bar{S}$ is the average value of source S over the control volume. It is a very attractive feature of the finite volume method that the discretised equation has a clear physical interpretation. Equation (4.4) states that the diffusive flux of ϕ leaving the east face minus the diffusive flux of ϕ entering the west face is equal to the generation of ϕ, i.e. it constitutes a balance equation for ϕ over the control volume.

In order to derive useful forms of the discretised equations, the interface diffusion coefficient Γ and the gradient $d\phi/dx$ at east ('e') and west ('w') are required. Following well-established practice, the values of the property ϕ and the diffusion coefficient are defined and evaluated at nodal points. To calculate gradients (and hence fluxes) at the control volume faces an approximate distribution of properties between nodal points is used. Linear approximations seem to be the obvious and simplest way of calculating interface values and the gradients. This practice is called central differencing (see Appendix A). In a uniform grid linearly interpolated values for Γ_e and Γ_w are given by

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2} \tag{4.5a}$$

$$\Gamma_e = \frac{\Gamma_P + \Gamma_E}{2} \tag{4.5b}$$

And the diffusive flux terms are evaluated as

$$\left(\Gamma A \frac{d\phi}{dx}\right)_e = \Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) \tag{4.6}$$

$$\left(\Gamma A \frac{d\phi}{dx}\right)_w = \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) \tag{4.7}$$

In practical situations, as illustrated later, the source term S may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:

$$\bar{S}\Delta V = S_u + S_p \phi_P \tag{4.8}$$

Substitution of equations (4.6), (4.7) and (4.8) into equation (4.4) gives

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) + (S_u + S_p \phi_P) = 0 \tag{4.9}$$

This can be re-arranged as

$$\boxed{\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_p\right)\phi_P = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w\right)\phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e\right)\phi_E + S_u} \tag{4.10}$$

Identifying the coefficients of ϕ_W and ϕ_E in equation (4.10) as a_W and a_E, and the coefficient of ϕ_P as a_P, the above equation can be written as

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u \tag{4.11}$$

where

a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}}A_w$	$\frac{\Gamma_e}{\delta x_{PE}}A_e$	$a_W + a_E - S_P$

The values of S_u and S_p can be obtained from the source model (4.8): $\bar{S}\Delta V = S_u + S_p\phi_P$. Equations (4.11) and (4.8) represent the discretised form of the equation (4.1). This type of discretised equation is central to all further developments.

Step 3: Solution of equations

Discretised equations of the form (4.11) must be set up at each of the nodal points in order to solve a problem. For control volumes that are adjacent to the domain boundaries the general discretised equation (4.11) is modified to incorporate boundary conditions. The resulting system of linear algebraic equations is then solved to obtain the distribution of the property ϕ at nodal points. Any suitable matrix solution technique may be enlisted for this task. In Chapter 7 we describe matrix solution methods that are specially designed for CFD procedures. The techniques of dealing with different types of boundary conditions will be examined in detail in Chapter 9.

4.3 Worked examples: one-dimensional steady state diffusion

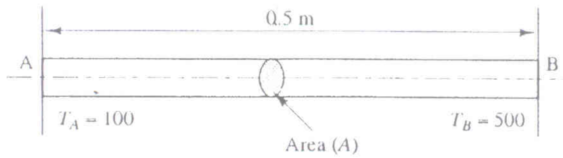
The application of the finite volume method to the solution of simple diffusion problems involving conductive heat transfer is presented in this section. The equation governing one-dimensional steady state conductive heat transfer is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + S = 0 \tag{4.12}$$

where thermal conductivity k takes the place of Γ in equation (4.3) and the dependent variable is temperature T. The source term can, for example, be heat generation due to an electrical current passing through the rod. The incorporation of boundary conditions as well as the treatment of source terms will be introduced by means of three worked examples.

Example 4.1 Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100 °C and 500 °C respectively. The one-

Fig. 4.3



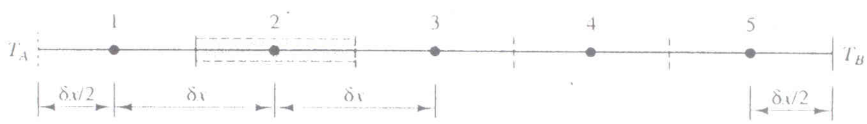
dimensional problem sketched in Figure 4.3 is governed by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{4.13}$$

Calculate the steady state temperature distribution in the rod. Thermal conductivity k equals 1000 W/m/K, cross-sectional area A is 10×10^{-3} m^2.

Solution Let us divide the length of the rod into five equal control volumes as shown in Figure 4.4. This gives $\delta x = 0.1$ m.

Fig. 4.4 The grid used



The grid consists of five nodes. For each one of nodes 2, 3 and 4 temperature values to the east and west are available as nodal values. Consequently, discretised equations of the form (4.10) can be readily written for control volumes surrounding these nodes:

$$\left(\frac{k_e}{\delta x_{PE}}A_e + \frac{k_w}{\delta x_{WP}}A_w\right)T_P = \left(\frac{k_w}{\delta x_{WP}}A_w\right)T_W + \left(\frac{k_e}{\delta x_{PE}}A_e\right)T_E \tag{4.14}$$

The thermal conductivity ($k_e = k_w = k$), node spacing (δx) and cross-sectional area ($A_e = A_w = A$) are constants. Therefore the **discretised equation for nodal points 2, 3 and 4** is

$$\boxed{a_P T_P = a_W T_W + a_E T_E} \tag{4.15}$$

with

a_W	a_E	a_P
$\frac{k}{\delta x}A$	$\frac{k}{\delta x}A$	$a_W + a_E$

S_u and S_p are zero in this case since there is no source term in the governing equation (4.13).

Nodes 1 and 5 are boundary nodes, and therefore require special attention. Integration of equation (4.13) over the control volume surrounding point 1 gives

$$kA\left(\frac{T_E - T_P}{\delta x}\right) - kA\left(\frac{T_P - T_A}{\delta x/2}\right) = 0 \tag{4.16}$$

This expression shows that the flux through control volume boundary A has been approximated by assuming a linear relationship between temperatures at boundary point A and node P. We can re-arrange (4.16) as follows:

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = 0.T_W + \left(\frac{k}{\delta x}A\right)T_E + \left(\frac{2k}{\delta x}A\right)T_A \tag{4.17}$$

Comparing equation (4.17) with equation (4.10) it can be easily identified that the fixed temperature boundary condition enters the calculation as a source term $(S_u + S_p T_P)$ with $S_u = (2kA/\delta x)T_A$ and $S_p = -2kA/\delta x$ and that the link to the (west) boundary side has been suppressed by setting coefficient a_W to zero.

Equation (4.17) may be cast in the same form as (4.11) to yield the **discretised equation for boundary node 1**:

$$a_P T_P = a_W T_W + a_E T_E + S_u \tag{4.18}$$

with

a_W	a_E	a_P	S_P	S_u
0	$\frac{kA}{\delta x}$	$a_W + a_E - S_p$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x}T_A$

The control volume surrounding node 5 can be treated in a similar manner. Its discretised equation is given by

$$kA\left(\frac{T_B - T_P}{\delta x/2}\right) - kA\left(\frac{T_P - T_W}{\delta x}\right) = 0 \tag{4.19}$$

As before we assume a linear temperature distribution between node P and boundary point B to approximate the heat flux through the control volume boundary. Equation (4.19) can be re-arranged as

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = \left(\frac{k}{\delta x}A\right)T_W + 0\,.\,T_E + \left(\frac{2k}{\delta x}A\right)T_B \tag{4.20}$$

The **discretised equation for boundary node 5** is

$$a_P T_P = a_W T_W + a_E T_E + S_u \tag{4.21}$$

where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_p$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x}T_B$

The discretisation process has yielded one equation for each of the nodal points 1 to 5. Substitution of numerical values gives $kA/\delta x = 100$ and the coefficients of each discretised equation can easily be worked out. Their values are given in Table 4.1.

Table 4.1

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	100	$200T_A$	−200	300
2	100	100	0	0	200
3	100	100	0	0	200
4	100	100	0	0	200
5	100	0	$200T_B$	−200	300

The resulting set of algebraic equations for this example is

$$\begin{aligned} 300T_1 &= 100T_2 + 200T_A \\ 200T_2 &= 100T_1 + 100T_3 \\ 200T_3 &= 100T_2 + 100T_4 \\ 200T_4 &= 100T_3 + 100T_5 \\ 300T_5 &= 100T_4 + 200T_B \end{aligned} \tag{4.22}$$

This set of equations can be re-arranged as

$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{bmatrix} \tag{4.23}$$

The above set of equations yields the steady state temperature distribution of the given situation. For simple problems involving a small number of nodes the resulting matrix equation can easily be solved with a software package such as MATLAB (*The Student Edition of MATLAB*, The Math Works Inc., 1992). For $T_A = 100$ and $T_B = 500$ the solution of (4.23) can obtained by using, for example, Gaussian elimination:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix} \tag{4.24}$$

The exact solution is a linear distribution between the specified boundary temperatures: $T = 800x + 100$. Figure 4.5 shows that the exact solution and the numerical results coincide.

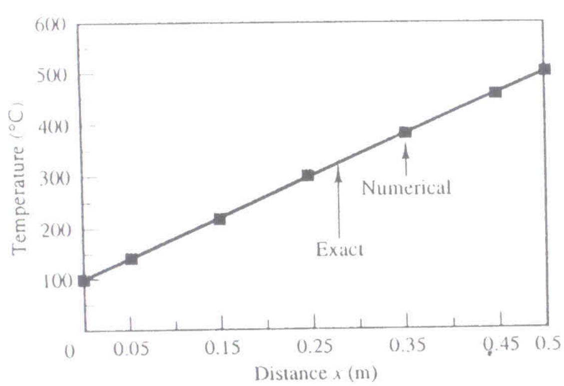


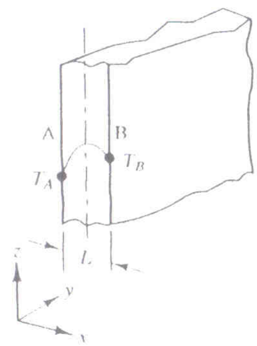
Fig. 4.5 Comparison of the numerical result with the analytical solution

Example 4.2 Now we discuss a problem that includes sources other than those arising from boundary conditions.

Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m/K and uniform heat generation $q = 1000$ kW/m^3. The faces A and B are at temperatures of 100 °C and 200 °C respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

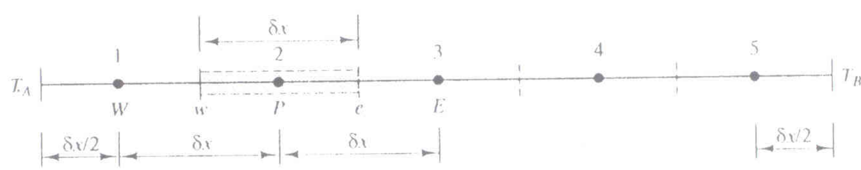
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + q = 0 \tag{4.25}$$

Fig. 4.6



Solution As before the method of solution is demonstrated using a simple grid. The domain is divided into five control volumes (see Figure 4.7) giving $\delta x = 0.004$ m; a unit area is considered in the y–z plane.

Fig. 4.7 The grid used



Formal integration of the governing equation over a control volume gives

$$\int_{\Delta V}\frac{d}{dx}\left(k\frac{dT}{dx}\right)dV + \int_{\Delta V} q\,dV = 0 \tag{4.26}$$

We treat the first term of the above equation as in the previous example. The second integral, the source term of the equation, is evaluated by calculating the average generation (i.e. $\bar{S}\Delta V = q\Delta V$) within each control volume. Equation (4.26) can be written as

$$\left[\left(kA\frac{dT}{dx}\right)_e - \left(kA\frac{dT}{dx}\right)_w\right] + q\Delta V = 0 \tag{4.27}$$

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_w A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.28}$$

The above equation can be re-arranged as

$$\left(\frac{k_e A}{\delta x} + \frac{k_w A}{\delta x}\right) T_P = \left(\frac{k_e A}{\delta x}\right) T_W + \left(\frac{k_w A}{\delta x}\right) T_E + qA\delta x \tag{4.29}$$

This equation is written in the general form of (4.11):

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.30}$$

Since $k_e = k_w = k$ we have the following coefficients:

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	$\frac{kA}{\delta x}$	$a_W + a_E - S_P$	0	$qA\delta x$

Equation (4.30) is valid for control volumes at **nodal points 2, 3 and 4**.

To incorporate the boundary conditions at nodes 1 and 5 we apply the linear approximation for temperatures between a boundary point and the adjacent nodal point. At node 1 the temperature at the west boundary is known. Integration of equation (4.25) at the control volume surrounding node 1 gives

$$\left[\left(kA\frac{dT}{dx}\right)_e - \left(kA\frac{dT}{dx}\right)_w\right] + q\Delta V = 0 \tag{4.31}$$

Introduction of the linear approximation for temperatures between A and P yields

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_A A\left(\frac{T_P - T_A}{\delta x/2}\right)\right] + qA\delta x = 0 \tag{4.32}$$

The above equation can be re-arranged, using $k_e = k_A = k$, to yield the discretised equation for **boundary node 1**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.33}$$

where

a_W	a_E	a_P	S_P	S_u
0	$\frac{kA}{\delta x}$	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x}T_A$

At nodal point 5, the temperature on the east face of the control volume is known. The node is treated in a similar way to boundary node 1. At boundary point 5 we have

$$\left[\left(k\,A\frac{dT}{dx}\right)_e - \left(k\,A\frac{dT}{dx}\right)_w\right] + q\Delta V = 0 \tag{4.34}$$

$$\left[k_B\,A\left(\frac{T_B - T_P}{\delta x/2}\right) - k_w\,A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.35}$$

The above equation can be re-arranged, noting that $k_B = k_w = k$, to give the discretised equation for **boundary node 5**:

$$a_P T_P = a_W T_W + a_E T_E + S_u \tag{4.36}$$

where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$q A \delta x + \frac{2kA}{\delta x} T_B$

Substitution of numerical values for $A = 1$, $k = 0.5$ W/m/K, $q = 1000$ kW/m^3 and $\delta x = 0.004$ m everywhere gives the coefficients of the discretised equations summarised in Table 4.2.

Table 4.2

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	125	$4000 + 250T_A$	-250	375
2	125	125	4000	0	250
3	125	125	4000	0	250
4	125	125	4000	0	250
5	125	0	$4000 + 250T_B$	-250	375

Given directly in matrix form the equations are

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 54000 \end{bmatrix} \tag{4.37}$$

The solution to the above set of equations is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix} \tag{4.38}$$

Comparison with the analytical solution

The analytical solution to this problem may be obtained by integrating equation (4.25) twice with respect to x and by subsequent application of the boundary conditions. This gives

$$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x) \right] x + T_A \tag{4.39}$$

The comparison between the finite volume solution and the exact solution is shown in Table 4.3 and Figure 4.8 and it can be seen that, even with a coarse grid of five nodes, the agreement is very good.

Table 4.3

Node number	1	2	3	4	5
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76

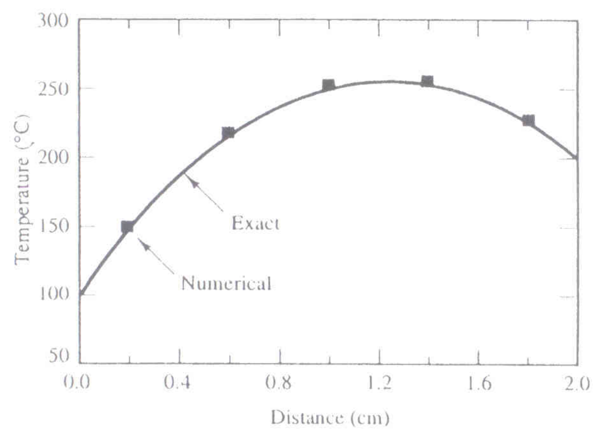


Fig. 4.8 Comparison of the numerical results with the analytical solution

Example 4.3 In the final worked example of this chapter we discuss the cooling of a circular fin by means of convective heat transfer along its length. Convection gives rise to a temperature-dependent heat loss or sink term in the governing equation.

Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100 °C (T_B) and the end is insulated. The fin is exposed to an ambient temperature of 20 °C. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \qquad (4.40)$$

where h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity of the material and T_∞ the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L - x)]}{\cosh(nL)} \qquad (4.41)$$

where $n^2 = hP/(kA)$, L is the length of the fin and x the distance along the fin. Data: $L = 1$ m, $hP/(kA) = 25\ m^{-2}$ (note kA is constant).

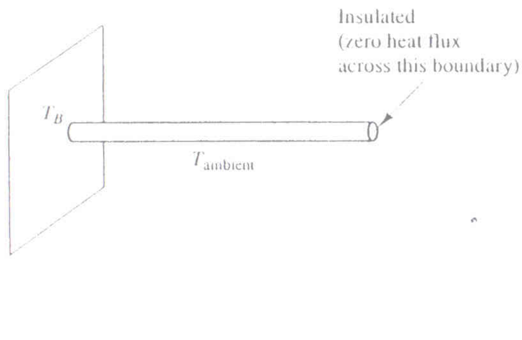


Fig. 4.9 The geometry for Example 4.3

Solution The governing equation in the example contains a sink term, $-hP(T - T_\infty)$, the convective heat loss, which is a function of the local temperature T. As before the first step in solving the problem by the finite volume method is to set up a grid. We use a uniform grid and divide the length into five control volumes so that $\delta x = 0.2$ m. The grid is shown in Figure 4.10.

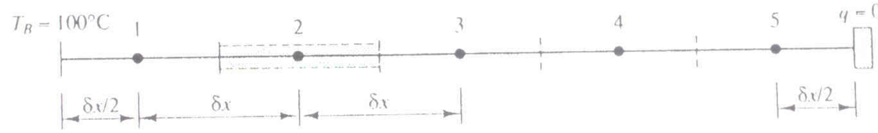


Fig. 4.10 The grid used in Example 4.3

When kA = constant, the governing equation (4.40) can be written as

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) - n^2(T - T_\infty) = 0 \text{ where } n^2 = hp/(kA) \tag{4.42}$$

Integration of the above equation over a control volume gives

$$\int_{\Delta V} \frac{d}{dx}\left(\frac{dT}{dx}\right) dV - \int_{\Delta V} n^2(T - T_\infty) dV = 0 \tag{4.43}$$

The first integral of the above equation is treated as in Examples 4.1 and 4.2; the second integral due to the source term in the equation is evaluated by assuming that the integral is locally constant within each control volume.

$$\left[\left(A\frac{dT}{dx}\right)_e - \left(A\frac{dT}{dx}\right)_w\right] - [n^2(T_P - T_\infty)A\delta x] = 0$$

First we develop a formula valid for nodal points 2, 3 and 4 by introducing the usual linear approximations for the temperature gradient. Subsequent division by cross-sectional area A gives

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_W}{\delta x}\right)\right] - [n^2(T_P - T_\infty)\delta x] = 0 \tag{4.44}$$

This can be re-arranged as

$$\left(\frac{1}{\delta x} + \frac{1}{\delta x}\right)T_P = \left(\frac{1}{\delta x}\right)T_W + \left(\frac{1}{\delta x}\right)T_E + n^2\delta x T_\infty - n^2\delta x T_P \tag{4.45}$$

For **interior nodal points 2, 3 and 4** we write, using general form (4.11),

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.46}$$

with

a_W	a_E	a_P	S_P	S_u
$\frac{1}{\delta x}$	$\frac{1}{\delta x}$	$a_W + a_E - S_P$	$-n^2\delta x$	$n^2\delta x T_\infty$

Next we apply the boundary conditions at node points 1 and 5. At node 1 the west control volume boundary is kept at a specified temperature. It is treated in the same way as Example 4.1, i.e.

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_B}{\delta x/2}\right)\right] - \left[n^2(T_P - T_\infty)\delta x\right] = 0 \qquad (4.47)$$

The coefficients of the discretised equation at **boundary node 1** are

a_W	a_E	a_P	S_p	S_u
0	$\frac{1}{\delta x}$	$a_W + a_E - S_p$	$-n^2\delta x - \frac{2}{\delta x}$	$n^2\delta x T_\infty + \frac{2}{\delta x}T_B$

At node 5 the flux across the east boundary is zero since the east side of the control volume is an insulated boundary:

$$\left[0 - \left(\frac{T_P - T_W}{\delta x}\right)\right] - \left[n^2(T_P - T_\infty)\delta x\right] = 0 \qquad (4.48)$$

Hence the east coefficient is set to zero. There are no additional source terms associated with the zero flux boundary condition. The coefficients at **boundary node 5** are given by

a_W	a_E	a_P	S_p	S_u
$\frac{1}{\delta x}$	0	$a_W + a_E - S_p$	$-n^2\delta x$	$n^2\delta x T_\infty$

Substituting numerical values gives the coefficients in Table 4.4

Table 4.4

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	5	$100 + 10T_B$	−15	20
2	5	5	100	−5	15
3	5	5	100	−5	15
4	5	5	100	−5	15
5	5	0	100	−5	10

The matrix form of the equations set is

$$\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ -5 & 15 & -5 & 0 & 0 \\ 0 & -5 & 15 & -5 & 0 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} \qquad (4.49)$$

The solution to the above system is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 64.22 \\ 36.91 \\ 26.50 \\ 22.60 \\ 21.30 \end{bmatrix} \qquad (4.50)$$

Comparison with the analytical solution

Table 4.5 compares the finite volume solution with analytical expression (4.41). The maximum percentage error ((analytical solution − finite volume solution)/analytical solution) is around 6%. Given the coarseness of the grid used in the calculation the numerical solution is reasonably close to the exact solution.

Table 4.5

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage Error*
1	0.1	64.22	68.52	4.30	6.27
2	0.3	36.91	37.86	0.95	2.51
3	0.5	26.50	26.61	0.11	0.41
4	0.7	22.60	22.53	−0.07	−0.31
5	0.9	21.30	21.21	−0.09	−0.42

The numerical solution can be improved by employing a finer grid. Let us consider the same problem with the rod length subdivided into 10 control volumes. The derivation of the discretised equations is the same as before, but the numerical values of the coefficients and source terms are different owing to the smaller grid spacing of $\delta x = 0.1$ m. The comparison of results of the second calculation with the analytical solution is shown in Figure 4.11 and Table 4.6. The second numerical results show better agreement with the analytical solution; now the maximum deviation is only 2%.

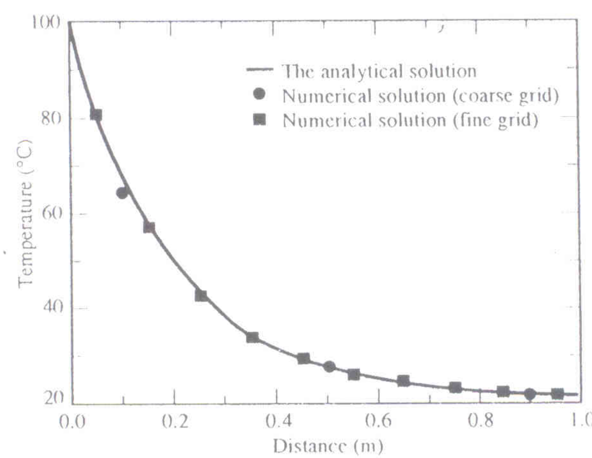


Fig. 4.11 Comparison of numerical and analytical results

Table 4.6

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage error*
1	0.05	80.59	82.31	1.72	2.08
2	0.15	56.94	57.79	0.85	1.47
3	0.25	42.53	42.93	0.40	0.93
4	0.35	33.74	33.92	0.18	0.53
5	0.45	28.40	28.46	0.06	0.21
6	0.55	25.16	25.17	0.01	0.03
7	0.65	23.21	23.19	−0.02	−0.08
8	0.75	22.06	22.03	−0.03	−0.13
9	0.85	21.47	21.39	−0.08	−0.37
10	0.95	21.13	21.11	−0.02	−0.09

4.4 Finite volume method for two-dimensional diffusion problems

The methodology used in deriving discretised equations in the one-dimensional case can be easily extended to two-dimensional problems. To illustrate the technique let us consider the two-dimensional steady state diffusion equation given by

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right)+S=0 \quad (4.51)$$

A portion of the two-dimensional grid used for the discretisation is shown in Figure 4.12.

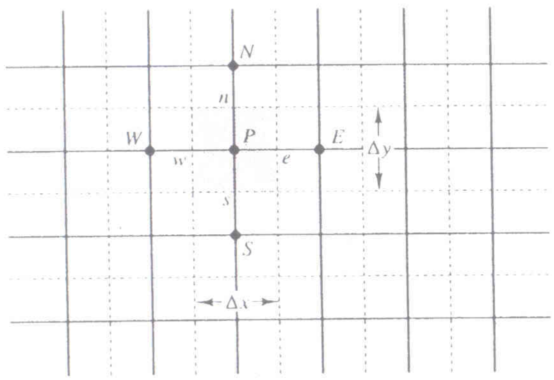


Fig. 4.12 A part of the two-dimensional grid

In addition to the east (E) and west (W) neighbours a general grid node P now also has north (N) and south (S) neighbours. The same notation as in the one-dimensional analysis is used for faces and cell dimensions. When the above equation is formally integrated over the control volume we obtain

$$\int_{\Delta V}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)dx\,.\,dy+\int_{\Delta V}\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right)dx\,.\,dy+\int_{\Delta V}S_\phi dV=0 \quad (4.52)$$

So, noting that $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$, we obtain:

$$\left[\Gamma_e A_e\left(\frac{\partial \phi}{\partial x}\right)_e-\Gamma_w A_w\left(\frac{\partial \phi}{\partial x}\right)_w\right]+\left[\Gamma_n A_n\left(\frac{\partial \phi}{\partial y}\right)_n-\Gamma_s A_s\left(\frac{\partial \phi}{\partial y}\right)_s\right]+S\Delta V=0 \quad (4.53)$$

As before this equation represents the balance of the generation of ϕ in a control volume and the fluxes through its cell faces. Using the approximations introduced in the previous section we can write expressions for the flux through control volume faces:

$$\text{Flux across the west face} = \Gamma_w A_w \left.\frac{\partial \phi}{\partial x}\right|_w = \Gamma_w A_w \frac{(\phi_P-\phi_W)}{\delta x_{WP}} \quad (4.54a)$$

$$\text{Flux across the east face} = \Gamma_e A_e \left.\frac{\partial \phi}{\partial x}\right|_e = \Gamma_e A_e \frac{(\phi_E-\phi_P)}{\delta x_{PE}} \quad (4.54b)$$

$$\text{Flux across the south face} = \Gamma_s A_s \left.\frac{\partial \phi}{\partial y}\right|_s = \Gamma_s A_s \frac{(\phi_P-\phi_S)}{\delta y_{SP}} \quad (4.54c)$$

$$\text{Flux across the north face} = \Gamma_n A_n \left.\frac{\partial \phi}{\partial y}\right|_n = \Gamma_n A_n \frac{(\phi_N-\phi_P)}{\delta y_{PN}} \quad (4.54d)$$

By substituting the above expressions into equation (4.53) we obtain

$$\Gamma_e A_e \frac{(\phi_E - \phi_P)}{\delta x_{PE}} - \Gamma_w A_w \frac{(\phi_P - \phi_W)}{\delta x_{WP}} + \Gamma_n A_n \frac{(\phi_N - \phi_P)}{\delta y_{PN}} - \Gamma_s A_s \frac{(\phi_P - \phi_S)}{\delta y_{SP}} + \bar{S}\Delta V = 0 \quad (4.55)$$

When the source term is represented in linearised form $\bar{S}\Delta V = S_u + S_p\phi_P$, this equation can be re-arranged as

$$\left(\frac{\Gamma_w A_w}{\delta x_{WP}} + \frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_s A_s}{\delta y_{SP}} + \frac{\Gamma_n A_n}{\delta y_{PN}} - S_p\right)\phi_P = \left(\frac{\Gamma_w A_w}{\delta x_{WP}}\right)\phi_W + \left(\frac{\Gamma_e A_e}{\delta x_{PE}}\right)\phi_E + \left(\frac{\Gamma_s A_s}{\delta y_{SP}}\right)\phi_S + \left(\frac{\Gamma_n A_n}{\delta y_{PN}}\right)\phi_N + S_u \quad (4.56)$$

Equation (4.56) is now cast in the general discretised equation form for interior nodes:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u \quad (4.57)$$

where

a_W	a_E	a_S	a_N	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$a_W + a_E + a_S + a_N - S_p$

The face areas in a two-dimensional case are $A_w = A_e = \Delta y$; $A_n = A_s = \Delta x$.

We obtain the distribution of the property ϕ in a given two-dimensional situation by writing discretised equations of the form (4.57) at each grid node of the subdivided domain. At the boundaries where the temperatures or fluxes are known the discretised equations are modified to incorporate boundary conditions in the manner demonstrated in Examples 4.1 and 4.2. The boundary side coefficient is set to zero (cutting the link with the boundary) and the flux crossing the boundary is introduced as a source which is appended to any existing S_u and S_p terms. Subsequently, the resulting set of equations is solved to obtain the two-dimensional distribution of the property ϕ. Example 7.2 in Chapter 7 shows how the method can be applied to calculate conductive heat transfer in two-dimensional situations.

4.5 Finite volume method for three-dimensional diffusion problems

Steady state diffusion in a three-dimensional situation is governed by

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right) + S = 0 \quad (4.58)$$

Now a three-dimensional grid is used to subdivide the domain. A typical control volume is shown in Figure 4.13.

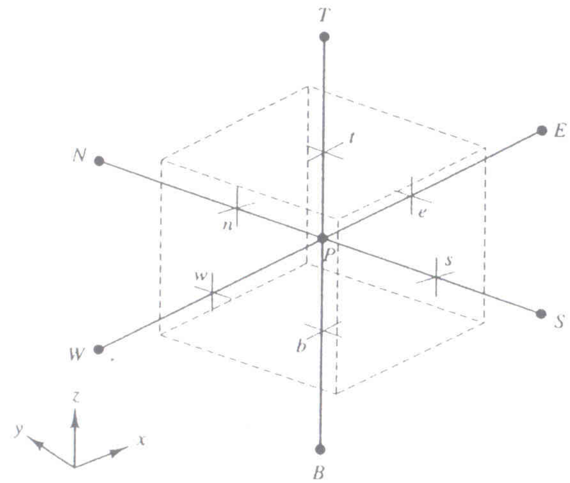


Fig. 4.13 A cell in three dimensions and neighbouring nodes

A cell containing node P now has six neighbouring nodes identified as west, east, south, north, bottom and top nodes (W, E, S, N, B, T). As before, the notation, w, e, s, n, b and t are used to refer to the west, east, south, north, bottom and top cell faces respectively.

Integration of Equation (4.58) over the control volume shown gives

$$\left[\Gamma_e A_e\left(\frac{\partial \phi}{\partial x}\right)_e - \Gamma_w A_w\left(\frac{\partial \phi}{\partial x}\right)_w\right] + \left[\Gamma_n A_n\left(\frac{\partial \phi}{\partial y}\right)_n - \Gamma_s A_s\left(\frac{\partial \phi}{\partial y}\right)_s\right] + \left[\Gamma_t A_t\left(\frac{\partial \phi}{\partial z}\right)_t - \Gamma_b A_b\left(\frac{\partial \phi}{\partial z}\right)_b\right] + \bar{S}\Delta V = 0 \tag{4.59}$$

Following the procedure developed for one- and two-dimensional cases the discretised form of the equation (4.59) is obtained:

$$\left[\Gamma_e \frac{(\phi_E - \phi_P)A_e}{\delta x_{PE}} - \Gamma_w \frac{(\phi_P - \phi_W)A_w}{\delta x_{WP}}\right] + \left[\Gamma_n \frac{(\phi_N - \phi_P)A_n}{\delta y_{PN}} - \Gamma_s \frac{(\phi_P - \phi_S)A_s}{\delta y_{SP}}\right] + \left[\Gamma_t \frac{(\phi_T - \phi_P)A_t}{\delta z_{PT}} - \Gamma_b \frac{(\phi_P - \phi_B)A_b}{\delta z_{BP}}\right] + (S_u + S_p\phi_P) = 0 \tag{4.60}$$

As before this can be re-arranged as to give the discretised equation for interior nodes:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + a_B\phi_B + a_T\phi_T + S_u \tag{4.61}$$

where

a_W	a_E	a_S	a_N	a_B	a_T	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta z_{PT}}$	$a_W + a_E + a_S + a_N + a_B + a_T - S_p$

Boundary conditions can be introduced by cutting links with the appropriate face(s) and modifying the source term as described in section 4.3.

4.6 Summary of discretised equations for diffusion problems

- The discretised equations for one-, two- and three-dimensional diffusion problems have been found to take the following general form:

$$\boxed{a_P\phi_P = \sum a_{nb}\phi_{nb} + S_u} \tag{4.62}$$

where Σ indicates summation over all neighbouring nodes (nb), and a_{nb} are the neighbouring coefficients, a_W, a_E in 1D, a_W, a_E, a_S, a_N in 2D and $a_W, a_E, a_S, a_N, a_B, a_T$ in 3D; ϕ_{nb} are the values of the property ϕ at the neighbouring nodes and $(S_u + S_p\phi_p)$ is the linearised source term.

- In all cases the coefficients around point P satisfy the following relation:

$$\boxed{a_P = \sum a_{nb} - S_p} \tag{4.63}$$

- A summary of the neighbour coefficients for one-, two- and three-dimensional diffusion problems is given in Table 4.7.

Table 4.7

	a_W	a_E	a_S	a_N	a_B	a_T
1D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	–	–	–	–
2D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	–	–
3D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta y_{BP}}$	$\frac{\Gamma_t A_t}{\delta z_{PT}}$

- Source terms can be included by identifying their linearised form $\bar{S}\Delta V = S_u + S_p\phi_P$ and specifying values for S_u and S_p.
- Boundary conditions are incorporated by cutting the link to the boundary side and introducing the boundary side flux – exact or linearly approximated through additional source terms S_u and S_p. For a one-dimensional control volume of width $\Delta\zeta$ with a boundary B:

link cutting : set coefficient $a_B = 0$ (4.64)

source contributions : fixed value ϕ_B : $S_u = \dfrac{2k_B A_B}{\Delta\zeta}\phi_B$;

$$S_p = -\frac{2k_B A_B}{\Delta\zeta} \tag{4.65}$$

fixed flux q_B : $S_u + S_p\phi_P = q_B$ (4.66)