Shear Failure Envelope of Hoek-Brown Criterion for Rockmass

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Abstract — *The Mohr-Coulomb (MC) criterion possesses a linear shear failure envelope and is widely used in geotechnical engineering calculations. However, the failure process in rockmass is more complex than that of soils, and it can be more adequately described by the Hoek-Brown (HB) failure criterion. This criterion has been updated as well as modified, but its effect on the shear failure envelope has not been studied. The shear failure envelope for the modified form of HB criterion is derived in this paper, which shows that a change in the value of parameter a from 0.5 to 0.6 affects the shear envelope more significantly for a rockmass characterized by a larger value of parameter m and high effective normal stress. The instantaneous friction angle is also affected by a change in the value of parameter a.*

Key Words: Failure criterion; Hoek-Brown criterion; Mohr envelope; Rockmass; Rock Mechanics; Shear envelope; Slip zone; Slope stability.

Introduction

Most of the theoretical development in geotechnical engineering has been based on the MC failure criterion. While it applies fairly well to soil mechanics problems, its extension to the problems of rock mechanics is questionable, although it is still applied in rock mechanics (Jumikis 1983; Daemen 1983). Based on available experimental evidence and theoretical experience with the fracture of rock, Hoek and Brown (1980) derived a criterion for its interpretation. This criterion has since been updated (Hoek and Brown 1988) as well as modified (Hoek et al. 1992).

While the shear failure envelope for the original form of HB criterion is known (Ucar 1986), the shear failure envelope for its modified form is not known. This paper presents its complete derivation. Two computational aids are given for quick use of the associated equations in practical calculations, and their use is illustrated. Some sample calculations are done in order to study the consequences of modifications in the original form of the HB criterion.

Construction of Shear Envelope

A strength criterion is essentially a function of stresses and some material parameters which, when satisfied, defines the material failure under the particular combination of stresses and for the given values of material parameters. In problems of geotechnical engineering, failure is characterized by shear deformation, and therefore it is more informative to express the strength criterion in terms of the normal and shear stresses acting on a plane inclined at some angle to the principal stress direction. This alternate representation is termed the shear envelope. A general procedure for derivation of the shear envelope was given by Balmer (1952). This procedure was then used by Ucar (1986) to derive the shear envelope of the original form of the HB criterion. The shear envelope for the generalized HB criterion is derived in this paper. The

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Notations

a	=	Material constant of HB criterion;
c	=	Cohesion in MC medium;
i	=	Effective roughness;
m	=	Material constant of HB criterion;
s	=	Material constant of HB criterion;
α	=	Angle between failure surface and direction of stress;
β	=	Instantaneous friction angle;
θ	=	Angular polar coordinate;
ϕ	=	Basic friction angle;
σ and τ	=	Normal and shear stress on a plane dipping at an angle α, respectively;
σ_c	=	Unconfined compressive strength of intact material;
σ_1 and σ_3	=	Major and minor principal stresses, respectively;
σ_r and σ_θ	=	Radial and tangential stress, respectively;
σ'	=	Derivative $\frac{\partial \sigma_1}{\partial \sigma_3}$;
τ'	=	Derivative $\frac{\partial \tau}{\partial \sigma}$

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procedure adopted does not require solution of any differential equation, and the solution given by Ucar (1986) now becomes a particular case of this derivation.

General Analysis

Let σ and τ be the normal and shear stresses, respectively, acting on a failure plane in a rockmass. The object of the analysis is to derive an expression for $\tau' = d\tau/d\sigma$ using the generalized HB criterion. From the analysis of Mohr circle, it can be easily shown that

$$\left[\sigma - \frac{\sigma_1 + \sigma_3}{2}\right]^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 \tag{1}$$

Let $\sigma' = \frac{\partial \sigma_1}{\partial \sigma_3}$ where σ_1 and σ_3 are the major and minor principal stresses, respectively.

By differentiating eq. 1 with respect to σ_3, the following expressions are obtained:

$$\sigma = \sigma_3 + \frac{\sigma_1 - \sigma_3}{1 + \sigma'} \tag{2}$$

$$\tau = \frac{\sigma_1 - \sigma_3}{1 + \sigma'}\sqrt{\sigma'} \tag{3}$$

$$\sigma_1 = \sigma + \tau\sqrt{\sigma'} \tag{4}$$

$$\sigma_3 = \sigma - \frac{\tau}{\sqrt{\sigma'}} \tag{5}$$

From eqs. 4 and 5, $$\sigma_1 - \sigma_3 = \tau\left[\frac{1 + \sigma'}{\sqrt{\sigma'}}\right] \tag{6}$$

Let α be the angle between the failure surface and the direction of minor principal stress and β be the inclination of envelope to the Mohr circle. From the Mohr circle construction (Fig. 1), it can be seen that $2\alpha = \beta + \pi/2$; therefore,

$$\tan\alpha = \sqrt{\sigma'} \text{ and } \tan\beta = \tau' \tag{7}$$

From eq. 7, $$\tan\beta = \frac{\tan^2\alpha - 1}{2\tan\alpha} \tag{8}$$

On substitution of eq. 7 into eq. 8, a quadratic equation in $\sqrt{\sigma'}$ is obtained. Taking the positive root of this equation and using eq. 8,

$$\sqrt{\sigma'} = \tau' + \sqrt{1 + (\tau')^2} = \frac{1 + \sin\beta}{\cos\beta} = \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) \tag{9}$$

Therefore,

$$\sigma' - 1 = \frac{2\sin\beta(1 + \sin\beta)}{\cos^2\beta} \text{ and } \sigma' + 1 = \frac{2(1 + \sin\beta)}{\cos^2\beta} \tag{10}$$

Application of Generalized HB criterion

The generalized HB criterion may be written as

$$\frac{\sigma_1 - \sigma_3}{\sigma_c} = \left\{m\frac{\sigma_3}{\sigma_c} + s\right\}^a \tag{11}$$

where σ_c is the unconfined compressive strength of the intact material and m, s and a are the material parameters associated with the HB criterion. Therefore,

$$\sigma' = \frac{\partial\sigma_1}{\partial\sigma_3} = 1 + ma\left(m\frac{\sigma_3}{\sigma_c} + s\right)^{a-1} \tag{12}$$

From eq. 2,

$$\frac{\sigma}{\sigma_c} = \frac{\sigma_3}{\sigma_c} + \frac{\left(m\frac{\sigma_3}{\sigma_c} + s\right)^a}{2 + ma\left(m\frac{\sigma_3}{\sigma_c} + s\right)^{a-1}} \tag{13}$$

$$\frac{\tau}{\sigma_c} = \frac{\left(m\frac{\sigma_3}{\sigma_c} + s\right)^a}{2 + ma\left(m\frac{\sigma_3}{\sigma_c} + s\right)^{a-1}}\left[1 + ma\left(m\frac{\sigma_3}{\sigma_c} + s\right)^{a-1}\right]^{1/2} \tag{14}$$

Combining eqs. 11 and 12,

$$\sigma' = 1 + \frac{ma}{\left(\frac{\sigma_1 - \sigma_3}{\sigma_c}\right)^{(a-1/a)}} \tag{15}$$

Substituting in eq. 15 from eq. 6,

$$\sigma' = 1 + \frac{ma}{\left[\frac{\tau}{\sigma_c}\left(\frac{1 + \sigma'}{\sqrt{\sigma'}}\right)\right]^{\left(\frac{1-a}{a}\right)}} \tag{16}$$

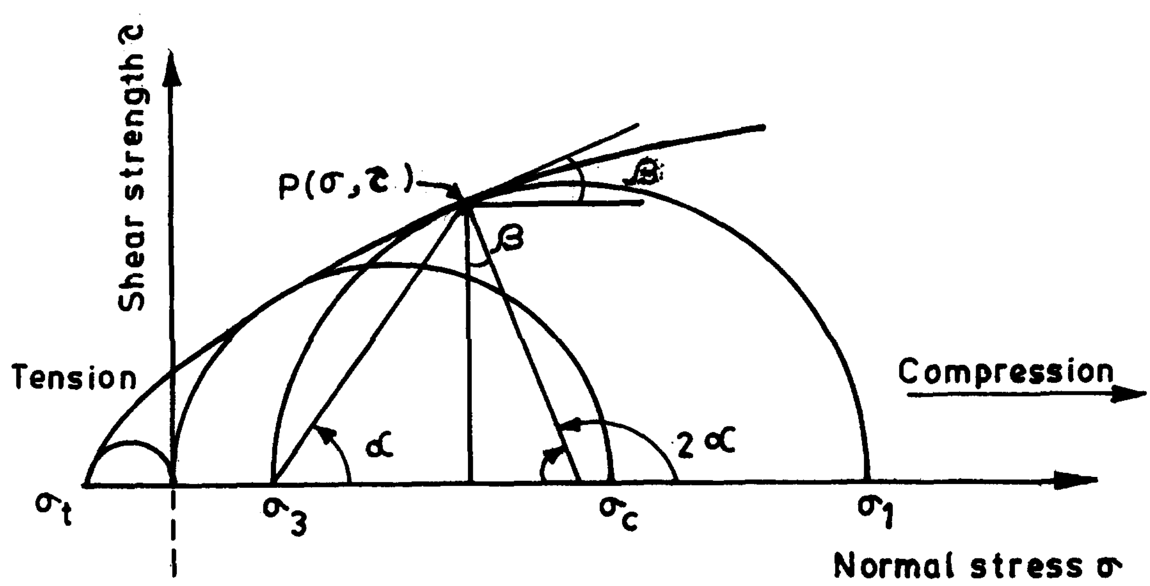
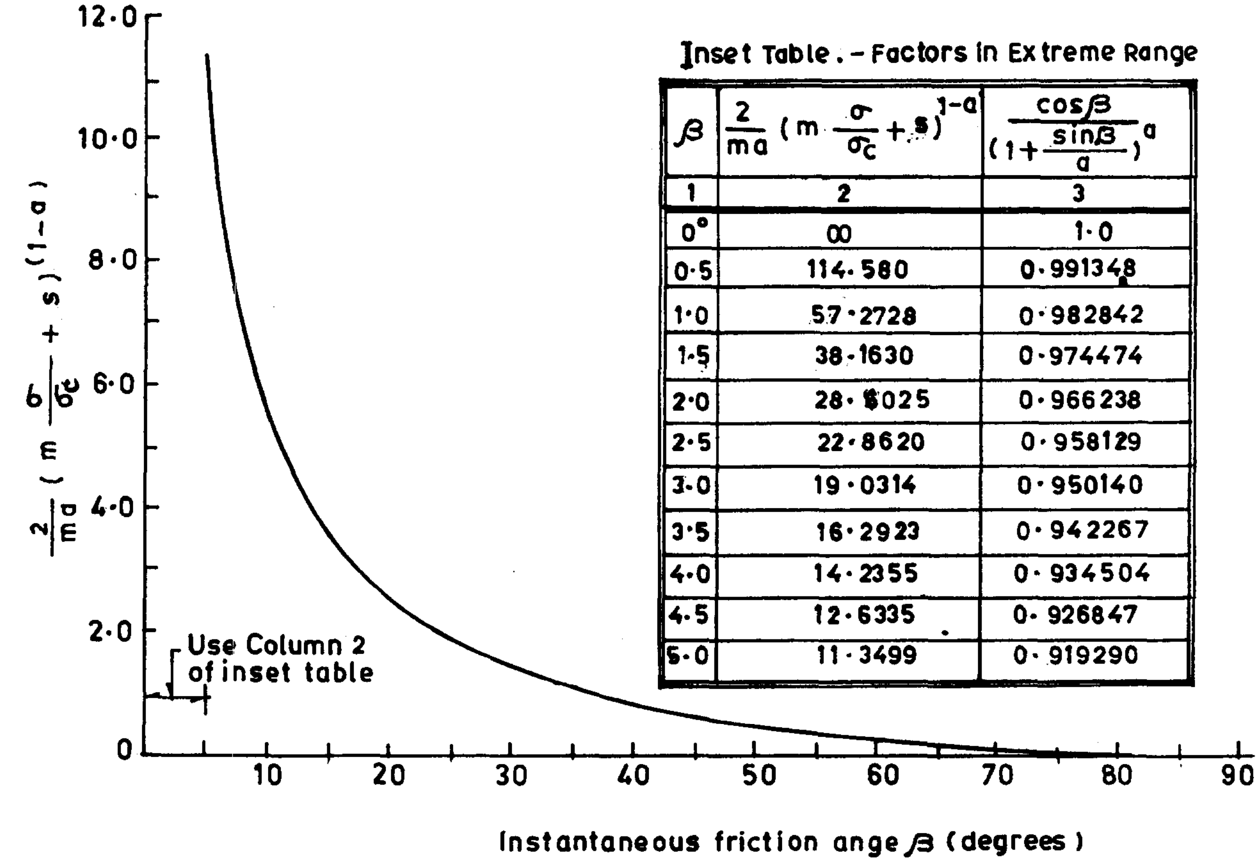


Figure 1. Graphical representation of stress conditions at failure.



Inset Table. - Factors in Extreme Range

β	$\frac{2}{ma}\left(m\frac{\sigma}{\sigma_c}+s\right)^{1-a}$	$\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^{a}}$
1	2	3
0°	∞	1·0
0·5	114·580	0·991348
1·0	57·2728	0·982842
1·5	38·1630	0·974474
2·0	28·6025	0·966238
2·5	22·8620	0·958129
3·0	19·0314	0·950140
3·5	16·2923	0·942267
4·0	14·2355	0·934504
4·5	12·6335	0·926847
5·0	11·3499	0·919290

Figure 2. Aid I for failure envelope computation.

With the help of eqs. 7, 8 and 9, eq. 16 may be written as

$$\frac{\tau}{\sigma_c}=\left(\frac{ma}{2}\right)^{\left(\frac{a}{1-a}\right)}\left(\frac{1-\sin\beta}{\sin\beta}\right)^{\left(\frac{a}{1-a}\right)}\left(\frac{\cos\beta}{2}\right) \tag{17}$$

From eqs. 6, 9 and 11,

$$2\,\frac{\tau}{\sigma_c}=\cos\beta\left(m\,\frac{\sigma_3}{\sigma_c}+s\right)^{a} \tag{18}$$

In eq. 18, σ_3 may be eliminated with the help of eq. 5. On using eqs. 7, 9 and 10, the final result appears as follows:

$$2\,\frac{\tau}{\sigma_c}=\cos\beta\left[m\left(\frac{\sigma}{\sigma_c}-\frac{\tau}{\sigma_c}\,\frac{\cos\beta}{1+\sin\beta}\right)+s\right]^{a} \tag{19}$$

Eliminating $\frac{\tau}{\sigma_c}$ from eq. 19 by using eq. 18 and on simplification, the following expression is obtained:

$$\frac{\sigma}{\sigma_c}=\frac{1}{m}\cdot\left(\frac{ma}{2}\right)^{\left(\frac{1}{1-a}\right)}\cdot\left(\frac{1-\sin\beta}{\sin\beta}\right)^{\left(\frac{1}{1-a}\right)}\cdot\left(1+\frac{\sin\beta}{a}\right)-\frac{s}{m} \tag{20}$$

The zero of the right hand side of eq. 20 occurs when $\sin\beta=1.0$ or $-a$. The sets of eqs. 13 and 14 or eqs. 17 and 20 give the required failure envelope. The latter set is in terms of angle β which is termed the instantaneous friction angle (Hoek, 1983). The failure envelope may also be presented in a compact form as follows:

$$2\,\frac{\tau}{\sigma_c}=\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^{a}}\cdot\left(m\,\frac{\sigma}{\sigma_c}+s\right)^{a} \tag{21}$$

For the particular case of $a=0.5$, the eqs. 17, 20 and 21 may be presented as

$$\frac{\tau}{\sigma_c}=\frac{m}{8}\cdot\frac{1-\sin\beta}{\sin\beta}\cdot\cos\beta=\frac{m}{8}\left(\cot\beta-\cos\beta\right) \tag{22}$$

$$\frac{\sigma}{\sigma_c}=\frac{m}{16}\cdot\left(\frac{1}{\sin^2\beta}+2\sin\beta\right)-\left(\frac{3m}{16}+\frac{s}{m}\right) \tag{23}$$

$$2\,\frac{\tau}{\sigma_c}=\frac{\cos\beta}{\sqrt{1+2\sin\beta}}\cdot\sqrt{m\,\frac{\sigma}{\sigma_c}+s} \tag{24}$$

Eqs. 22 and 23 were also derived by Ucar (1986), although a longer procedure was used in which solution of a differential equation was also involved.

Computation Procedure

The solution procedure of failure envelope than becomes:

Step I: For the given value of $\frac{\sigma}{\sigma_c}$, m, s and a, solve eq. 20 for β.

Step II: Compute value of the factor $\left[\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^{a}}\right]$

Step III: Use eq. 21 to obtain $\frac{\tau}{\sigma_c}$

The eq. 13 may also be written as

$$\frac{\sigma}{\sigma_c}=\frac{\sigma_3}{\sigma_c}+\frac{\left(\frac{\sigma_1-\sigma_3}{\sigma_c}\right)}{2+ma\cdot\left(\frac{\sigma_1-\sigma_3}{\sigma_c}\right)^{\left(\frac{a-1}{a}\right)}} \tag{25}$$

Eq. 25 may also be used for computation of σ/σ_c to be used in step I above. To facilitate computations with the above failure envelope, two computation aids are given in Figures 2 and 3. The curves in these figures do not show any significant variation for various values of a between 0.5 and 0.6; therefore, they may be valid for all values of a in this range. Figure 2 is derived by recasting eq. 20 as follows:

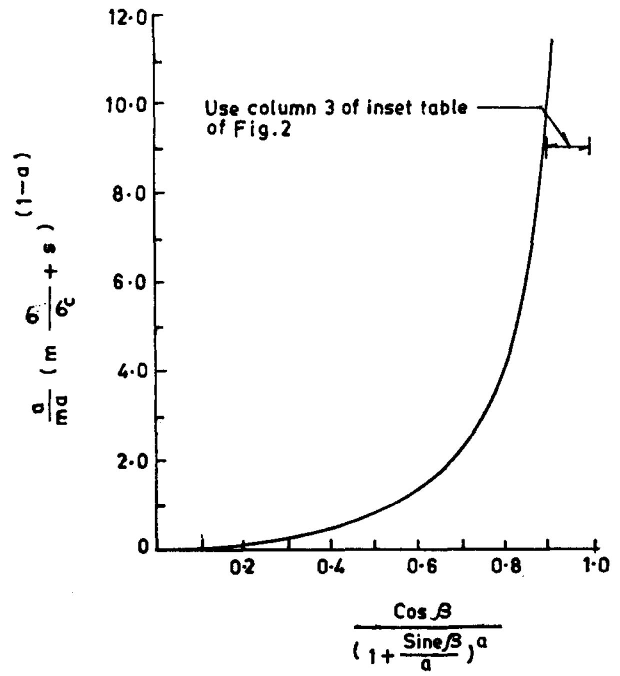


Figure 3. Aid 2 for failure envelope computation.

$$\frac{2}{ma}\cdot\left(m\frac{\sigma}{\sigma_c}+s\right)^{(1-a)}=\left(\frac{1-\sin\beta}{\sin\beta}\right)\cdot\left(1+\frac{\sin\beta}{a}\right)^{(1-a)} \quad (26)$$

Illustrative Computations

The theory presented in the previous sections was used to prepare sample illustrations of Figures 4 and 5, in which the failure envelope and variation friction angle are presented for extreme values of m, s and a. It was shown by Hoek (1983) that large values of m, of the order of 15–25, give a steeply inclined Mohr envelope and high instantaneous friction angle at low effective stress levels. These characteristics correspond to brittle behavior of igneous and metamorphic rocks such as andesite, gneiss and granite. The lower m values, of the order of 3–7, give a lower instantaneous friction angle, and tend to be associated with more ductile carbonate rocks such as limestone and dolomite. The parameter s is unity for intact rock and implies a finite tensile strength. The minimum value of s = 0 applies to a heavily jointed or broken rockmass. In such cases, the tensile strength is zero and the rockmass has zero cohesive strength when the effective normal stress is zero. Figures 4 and 5 may be examined in light of this information.

It is seen in Figure 4 that for m = 25, the envelope for a = 0.6 lies considerably above that for a = 0.5, suggesting a more brittle behavior. The curves for s = 0 lie a little below the corresponding s = 1 curves but otherwise these curves are close to each other. The envelopes behave differently for smaller values of m. A change in the value of s from unity to zero has a more pronounced effect on the failure envelope than it did in the case of larger values of m. Also, the difference in envelopes for a = 0.5 and 0.6 is significant at higher values of effective normal stress and s = 1.0. When s = 0, the two envelopes are practically identical.

Figure 5 shows that a change in the value of a from 0.5 to 0.6 enhances the friction angle, which becomes more pronounced for higher effective normal stress. The curves for s = 0 lie above the corresponding curves for s = 1. These curves differ more significantly for smaller values of effective normal stress. For higher values of m, the region of significant difference is narrow (0 to 0.1), while for smaller values of m, this region is considerably larger (0 to 0.6). This appears reasonable because larger values of m apply to brittle rocks, and smaller values to ductile rocks. Some of these findings may be substantiated as follows. Let ϕ be the effective friction angle and i the effective roughness. These are related as follows:

$$\tau=\sigma\tan(\phi+i) \quad (27)$$

Eq. 27 applies to the shear resistance of non-planar rock joints and predicts an increase in the instantaneous friction angle. In the case of sands, the angle i denotes the average angle of deviation of particle displacements from the direction of the applied shear stress. In the case of rock joints, i is related to the geometry of the asperities. The value of i reduces at high normal stress, as asperities would tend to shear off (Barton 1976). This feature is clearly visible in Figure 5.

At extremely low normal stress, Barton and Choubey (1977) have measured the instantaneous friction angle in excess of 88°. Barton (1976) has also discussed the possibility of having a vertical tangent at or close to the shear stress axis (zero normal stress), particularly so for all mating surfaces when there is measurable non-planarity (i.e. JRC > 5). Therefore, a theoretical calculation of β = 90.0 at σ = 0 (Fig. 5) for σ = 0 appears reasonable. These arguments may also be valid in the case of rockmass, particularly during the initial stages of loading when the elastic stress distribution is also valid.

Numerical Examples

Ucar (1986) has given three examples which are resolved subsequently to illustrate the use of Figures 2 and 3.

Example 1. Granite rockmass of good quality. Slightly weathered with joint spacing of 2 m.

Blocky seamy, folded and faulted. This rock description in conjunction with the table of constants given by Hoek (1996) gives the following: m = 2.5, s = 0.004, a = 0.5. Let σ/σ_c = 2.0.

Therefore, $\frac{2}{ma}\cdot\left(m\frac{\sigma}{\sigma_c}+s\right)^{(1-a)}=3.58$

From Figure 2, β = 15° and from Figure 3,

$$\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^a}=0.785$$

From eq. 21, $\frac{\tau}{\sigma_c}=0.878$

Example 2. Amphibolite of very poor quality, highly jointed and weathered with joint spacing less than 50 mm. With this rock description and the table of constants given by Hoek (1996), m = 0.025, s = 0, a=0.5, and let σ/σ_c = 1.0.

$$\frac{2}{ma}\cdot\left(m\frac{\sigma}{\sigma_c}+s\right)^{(1-a)}=80.0$$

From the inset table of Figure 2, β = 0.8° and

$$\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^a}$$

Therefore, eq. 21 gives $\frac{\tau}{\sigma_c}=0.247$

Example 3. Gneiss of poor quality with numerous weathered joints at 50- to 100-mm spacing. With this rock description and the table of constants given by Hoek (1996),

m = 0.13, $s = 10^{-5}$, a = 0.5. Let σ/σ_c = 5.0.

From the inset table of Figure 2, β = 2.33° and

$$\frac{\cos\beta}{\left(1+\frac{\sin\beta}{a}\right)^a}=0.961$$

From eq. 21, τ/σ_c = 0.3874.

These results were also obtained by Ucar (1986).

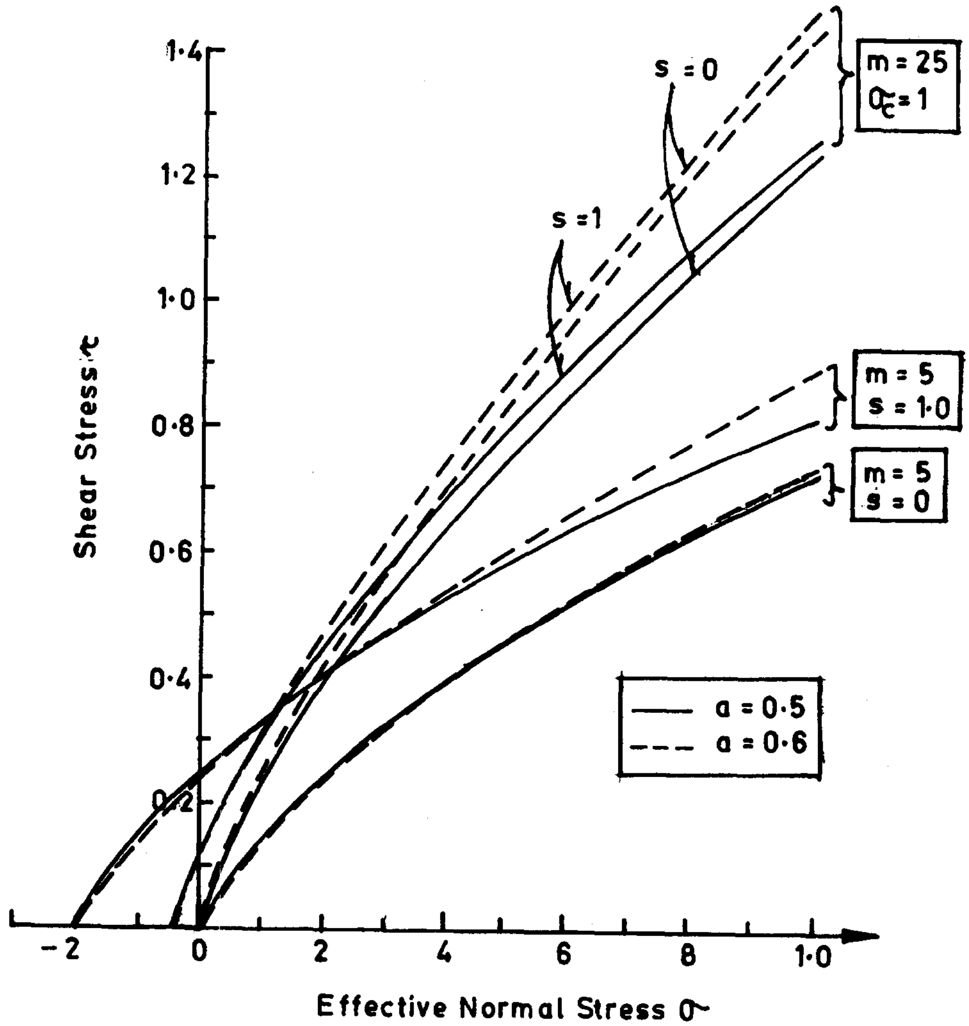


Figure 4. Failure envelopes for sample material parameter values.

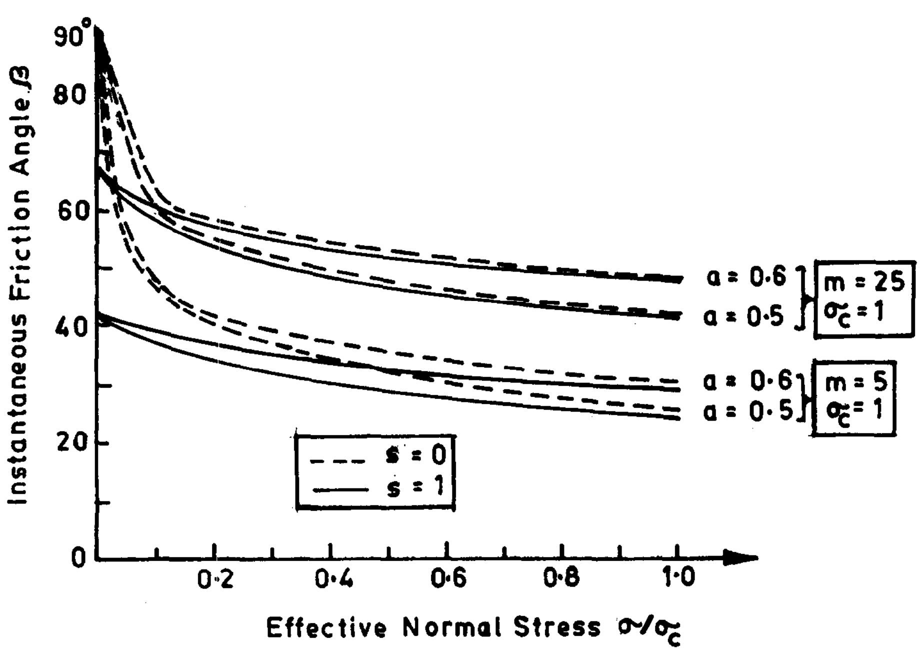


Figure 5. Instantaneous friction angle for sample material parameter values.

Applications

Aapplications of the above shear envelope in the computation of the slip line field around circular openings in a jointed Hoek-Brown medium (Kumar 1997) have been shown. The critical height of a drained slope in a Hoek-Brown medium containing a planar discontinuity (Kumar 1998) has also been computed. In both these applications, an iterative computational scheme has been developed to account for the nonlinear nature of the shear envelope. This scheme is not needed in the use of the Mohr-Coulomb criterion because its shear envelope is linear.

Concluding Remarks

The shear envelope for a generalized form of HB criterion is derived from which the information of available published literature can be obtained as a particular case.. The behavior of the shear envelope for typical values of material parameters is studied and examined in the light of the published information. It is found that by changing the value of parameter a in eq. 11, which is the main feature of the generalized HB criterion, both the shear envelope and the friction angle change. This change can easily be quantified from the theory presented in this paper.

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