Chapter 08.07
Finite Difference Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *Understand what the finite difference method is and how to use it to solve problems.*

What is the finite difference method?

The finite difference method is used to solve ordinary differential equations that have conditions imposed on the boundary rather than at the initial point. These problems are called boundary-value problems. In this chapter, we solve second-order ordinary differential equations of the form

$$\frac{d^2y}{dx^2} = f(x, y, y'), a \le x \le b,\tag{1}$$

with boundary conditions

$$y(a) = y_a \text{ and } y(b) = y_b\tag{2}$$

Many academics refer to boundary value problems as position-dependent and initial value problems as time-dependent. That is not necessarily the case as illustrated by the following examples.

The differential equation that governs the deflection y of a simply supported beam under uniformly distributed load (Figure 1) is given by

$$\frac{d^2y}{dx^2} = \frac{qx(L-x)}{2EI}\tag{3}$$

where

$x = $ location along the beam (in)
$E = $ Young's modulus of elasticity of the beam (psi)
$I = $ second moment of area (in^4)
$q = $ uniform loading intensity (lb/in)
$L = $ length of beam (in)

The conditions imposed to solve the differential equation are

$$y(x = 0) = 0\tag{4}$$
$$y(x = L) = 0$$

Clearly, these are boundary values and hence the problem is considered a boundary-value problem.
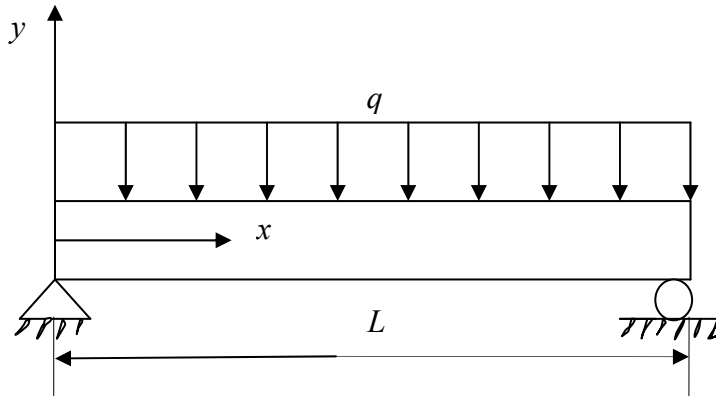
Figure 1 Simply supported beam with uniform distributed load.

Now consider the case of a cantilevered beam with a uniformly distributed load (Figure 2). The differential equation that governs the deflection y of the beam is given by

$$\frac{d^2y}{dx^2} = \frac{q(L-x)^2}{2EI}$$

(5)

where

x = location along the beam (in)
E = Young's modulus of elasticity of the beam (psi)
I = second moment of area (in^4)
q = uniform loading intensity (lb/in)
L = length of beam (in)

The conditions imposed to solve the differential equation are

$$y(x = 0) = 0$$

(6)

$$\frac{dy}{dx}(x = 0) = 0$$

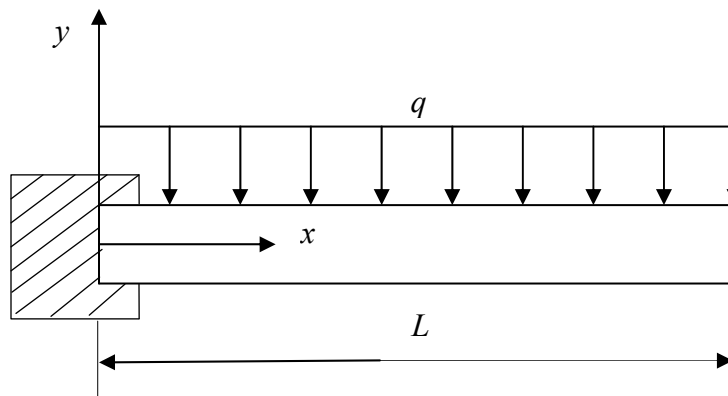Clearly, these are initial values and hence the problem needs to be considered as an initial value problem.

Figure 2 Cantilevered beam with a uniformly distributed load.

Example 1

The deflection y in a simply supported beam with a uniform load q and a tensile axial load T is given by

$$\frac{d^2 y}{dx^2} - \frac{Ty}{EI} = \frac{qx(L-x)}{2EI}$$ (E1.1)

where

   x = location along the beam (in)
   T = tension applied (lbs)
   E = Young's modulus of elasticity of the beam (psi)
   I = second moment of area (in^4)
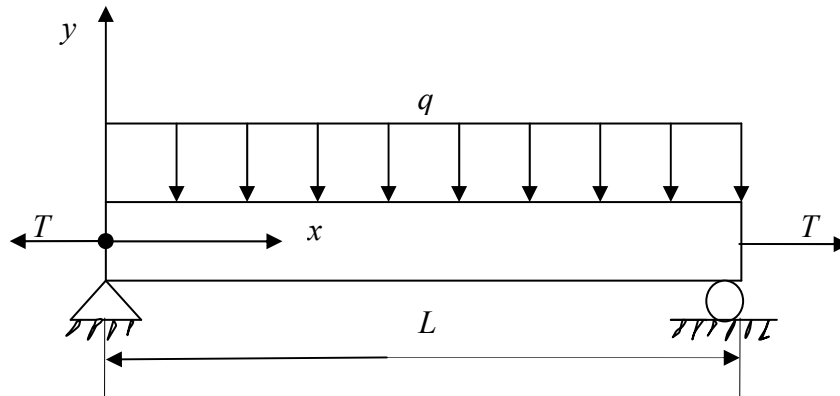 q = uniform loading intensity (lb/in)
   L = length of beam (in)

Figure 3 Simply supported beam for Example 1.

Given,

   $T = 7200$ lbs, $q = 5400$ lbs/in, $L = 75$ in, $E = 30$ Msi, and $I = 120$ in^4,

a) Find the deflection of the beam at $x = 50''$. Use a step size of $\Delta x = 25''$ and approximate the derivatives by central divided difference approximation.
b) Find the relative true error in the calculation of $y(50)$.

Solution

a) Substituting the given values,

$$\frac{d^2 y}{dx^2} - \frac{7200 y}{(30 \times 10^6)(120)} = \frac{(5400)x(75-x)}{2(30 \times 10^6)(120)}$$

$$\frac{d^2 y}{dx^2} - 2 \times 10^{-6} y = 7.5 \times 10^{-7} x(75-x)$$ (E1.2)

Approximating the derivative $\dfrac{d^2 y}{dx^2}$ at node i by the central divided difference approximation,
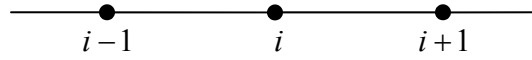
Figure 4 Illustration of finite difference nodes using central divided difference method.

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \qquad \text{(E1.3)}$$

We can rewrite the equation as

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} - 2 \times 10^{-6} y_i = 7.5 \times 10^{-7} x_i (75 - x_i) \qquad \text{(E1.4)}$$

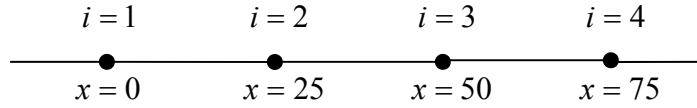Since $\Delta x = 25$, we have 4 nodes as given in Figure 3

Figure 5 Finite difference method from $x = 0$ to $x = 75$ with $\Delta x = 25$.

The location of the 4 nodes then is

$$x_0 = 0$$
$$x_1 = x_0 + \Delta x = 0 + 25 = 25$$
$$x_2 = x_1 + \Delta x = 25 + 25 = 50$$
$$x_3 = x_2 + \Delta x = 50 + 25 = 75$$

Writing the equation at each node, we get

<u>Node 1:</u> From the simply supported boundary condition at $x = 0$, we obtain

$$y_1 = 0 \qquad \text{(E1.5)}$$

<u>Node 2:</u> Rewriting equation (E1.4) for node 2 gives

$$\frac{y_3 - 2y_2 + y_1}{(25)^2} - 2 \times 10^{-6} y_2 = 7.5 \times 10^{-7} x_2 (75 - x_2)$$

$$0.0016 y_1 - 0.003202 y_2 + 0.0016 y_3 = 7.5 \times 10^{-7} (25)(75 - 25)$$

$$0.0016 y_1 - 0.003202 y_2 + 0.0016 y_3 = 9.375 \times 10^{-4} \qquad \text{(E1.6)}$$

<u>Node 3:</u> Rewriting equation (E1.4) for node 3 gives

$$\frac{y_4 - 2y_3 + y_2}{(25)^2} - 2 \times 10^{-6} y_3 = 7.5 \times 10^{-7} x_3 (75 - x_3)$$

$$0.0016 y_2 - 0.003202 y_3 + 0.0016 y_3 = 7.5 \times 10^{-7} (50)(75 - 50)$$

$$0.0016 y_2 - 0.003202 y_3 + 0.0016 y_3 = 9.375 \times 10^{-4} \qquad \text{(E1.7)}$$

<u>Node 4:</u> From the simply supported boundary condition at $x = 75$, we obtain

$$y_4 = 0 \qquad \text{(E1.8)}$$

Equations (E1.5-E1.8) are 4 simultaneous equations with 4 unknowns and can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0016 & -0.003202 & 0.0016 & 0 \\ 0 & 0.0016 & -0.003202 & 0.0016 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.375 \times 10^{-4} \\ 9.375 \times 10^{-4} \\ 0 \end{bmatrix}$$

The above equations have a coefficient matrix that is tridiagonal (we can use Thomas' algorithm to solve the equations) and is also strictly diagonally dominant (convergence is guaranteed if we use iterative methods such as the Gauss-Siedel method). Solving the equations we get,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5852 \\ -0.5852 \\ 0 \end{bmatrix}$$

$$y(50) = y(x_2) \approx y_2 = -0.5852"$$

The exact solution of the ordinary differential equation is derived as follows. The homogeneous part of the solution is given by solving the characteristic equation

$$m^2 - 2 \times 10^{-6} = 0$$
$$m = \pm 0.0014142$$

Therefore,

$$y_h = K_1 e^{0.0014142x} + K_2 e^{-0.0014142x}$$

The particular part of the solution is given by

$$y_p = Ax^2 + Bx + C$$

Substituting the differential equation (E1.2) gives

$$\frac{d^2 y_p}{dx^2} - 2 \times 10^{-6} y_p = 7.5 \times 10^{-7} x(75 - x)$$

$$\frac{d^2}{dx^2}(Ax^2 + Bx + C) - 2 \times 10^{-6}(Ax^2 + Bx + C) = 7.5 \times 10^{-7} x(75 - x)$$

$$2A - 2 \times 10^{-6}(Ax^2 + Bx + C) = 7.5 \times 10^{-7} x(75 - x)$$

$$-2 \times 10^{-6} Ax^2 - 2 \times 10^{-6} Bx + (2A - 2 \times 10^{-6} C) = 5.625 \times 10^{-5} x - 7.5 \times 10^{-7} x^2$$

Equating terms gives

$$-2 \times 10^{-6} A = -7.5 \times 10^{-7}$$
$$-2 \times 10^{-6} B = -5.625 \times 10^{-5}$$
$$2A - 2 \times 10^{-6} C = 0$$

Solving the above equation gives

$$A = 0.375$$
$$B = -28.125$$
$$C = 3.75 \times 10^5$$

The particular solution then is
$$y_p = 0.375x^2 - 28.125x + 3.75 \times 10^5$$
The complete solution is then given by
$$y = 0.375x^2 - 28.125x + 3.75 \times 10^5 + K_1 e^{0.0014142x} + K_2 e^{-0.0014142x}$$
Applying the following boundary conditions
$$y(x = 0) = 0$$
$$y(x = 75) = 0$$
we obtain the following system of equations
$$K_1 + K_2 = -3.75 \times 10^5$$
$$1.1119K_1 + 0.89937K_2 = -3.75 \times 10^5$$
These equations are represented in matrix form by
$$\begin{bmatrix} 1 & 1 \\ 1.1119 & 0.89937 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -3.75 \times 10^5 \\ -3.75 \times 10^5 \end{bmatrix}$$
A number of different numerical methods may be utilized to solve this system of equations such as the Gaussian elimination. Using any of these methods yields
$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -1.775656226 \times 10^5 \\ -1.974343774 \times 10^5 \end{bmatrix}$$
Substituting these values back into the equation gives
$$y = 0.375x^2 - 28.125x + 3.75 \times 10^5 - 1.775656266 \times 10^5 e^{0.0014142x} - 1.974343774 \times 10^5 e^{-0.0014142x}$$
Unlike other examples in this chapter and in the book, the above expression for the deflection of the beam is displayed with a larger number of significant digits. This is done to minimize the round-off error because the above expression involves subtraction of large numbers that are close to each other.

b) To calculate the relative true error, we must first calculate the value of the exact solution at $y = 50$.
$$y(50) = 0.375(50)^2 - 28.125(50) + 3.75 \times 10^5 - 1.775656266 \times 10^5 e^{0.0014142(50)}$$
$$- 1.974343774 \times 10^5 e^{-0.0014142(50)}$$
$$y(50) = -0.5320$$
The true error is given by
$$E_t = \text{Exact Value} - \text{Approximate Value}$$
$$E_t = -0.5320 - (-0.5852)$$
$$E_t = 0.05320$$
The relative true error is given by
$$\in_t = \frac{\text{True Error}}{\text{True Value}} \times 100\%$$
$$\in_t = \frac{0.05320}{-0.5320} \times 100\%$$
$$\in_t = -10\%$$

Example 2

Take the case of a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. For a thick pressure vessel of inner radius a and outer radius b, the differential equation for the radial displacement u of a point along the thickness is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{E2.3}$$

The inner radius $a = 5''$ and the outer radius $b = 8''$, and the material of the pressure vessel is ASTM A36 steel. The yield strength of this type of steel is 36 ksi. Two strain gages that are bonded tangentially at the inner and the outer radius measure normal tangential strain as

$$\in_{t/r=a} = 0.00077462$$

$$\in_{t/r=b} = 0.00038462 \tag{E2.4a,b}$$

at the maximum needed pressure. Since the radial displacement and tangential strain are related simply by

$$\in_t = \frac{u}{r}, \tag{E2.5}$$

then

$$u\big|_{r=a} = 0.00077462 \times 5 = 0.0038731''$$

$$u\big|_{r=b} = 0.00038462 \times 8 = 0.0030769''$$

The maximum normal stress in the pressure vessel is at the inner radius $r = a$ and is given by

$$\sigma_{max} = \frac{E}{1-v^2}\left(\frac{u}{r}\bigg|_{r=a} + v\frac{du}{dr}\bigg|_{r=a}\right) \tag{E2.7}$$

where

$\quad E =$ Young's modulus of steel (E= 30 Msi)
$\quad v =$ Poisson's ratio ($v = 0.3$)

The factor of safety, FS is given by

$$FS = \frac{\text{Yield strength of steel}}{\sigma_{max}} \tag{E2.8}$$

   a) Divide the radial thickness of the pressure vessel into 6 equidistant nodes, and find the radial displacement profile

   b) Find the maximum normal stress and factor of safety as given by equation (E2.8)

   c) Find the exact value of the maximum normal stress as given by equation (E2.8) if it is given that the exact expression for radial displacement is of the form

$$u = C_1 r + \frac{C_2}{r}.$$

      Calculate the relative true error.

Solution
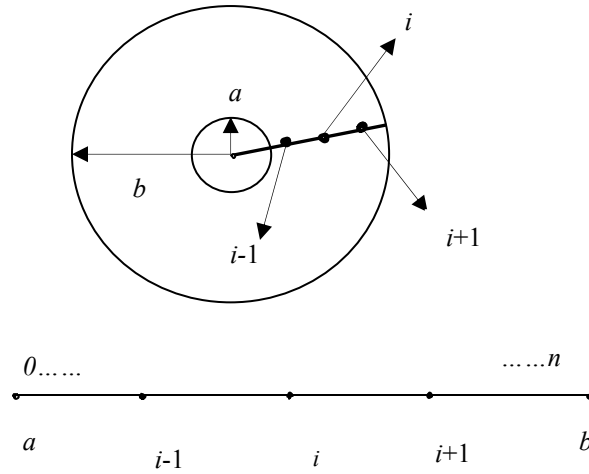
Figure 4 Nodes along the radial direction.

a) The radial locations from $r = a$ to $r = b$ are divided into n equally spaced segments, and hence resulting in $n+1$ nodes. This will allow us to find the dependent variable u numerically at these nodes.

At node i along the radial thickness of the pressure vessel,

$$\frac{d^2 u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \tag{E2.9}$$

$$\frac{du}{dr} \approx \frac{u_{i+1} - u_i}{\Delta r} \tag{E2.10}$$

Such substitutions will convert the ordinary differential equation into a linear equation (but with more than one unknown). By writing the resulting linear equation at different points at which the ordinary differential equation is valid, we get simultaneous linear equations that can be solved by using techniques such as Gaussian elimination, the Gauss-Siedel method, etc.

Substituting these approximations from Equations (E2.9) and (E2.10) in Equation (E2.3)

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i} \frac{u_{i+1} - u_i}{\Delta r} - \frac{u_i}{r_i^2} = 0 \tag{E2.11}$$

$$\left(\frac{1}{(\Delta r)^2} + \frac{1}{r_i \Delta r} \right) u_{i+1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i \Delta r} - \frac{1}{r_i^2} \right) u_i + \frac{1}{(\Delta r)^2} u_{i-1} = 0 \tag{E2.12}$$

Let us break the thickness, $b - a$, of the pressure vessel into $n+1$ nodes, that is $r = a$ is node $i = 0$ and $r = b$ is node $i = n$. That means we have $n+1$ unknowns.

We can write the above equation for nodes $1, ..., n-1$. This will give us $n-1$ equations. At the edge nodes, $i = 0$ and $i = n$, we use the boundary conditions of

$$u_0 = u\big|_{r=a}$$

$$u_n = u\big|_{r=b}$$

This gives a total of $n+1$ equations. So we have $n+1$ unknowns and $n+1$ linear equations. These can be solved by any of the numerical methods used for solving simultaneous linear equations.

We have been asked to do the calculations for $n = 5$, that is a total of 6 nodes. This gives

$$\Delta r = \frac{b-a}{n}$$

$$= \frac{8-5}{5}$$

$$= 0.6\text{"}$$

At node $i = 0, r_0 = a = 5\text{"}, \ u_0 = 0.0038731\text{"}$ (E2.13)

At node $i = 1, r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6\text{"}$ (E2.14)

$$\frac{1}{0.6^2}u_0 + \left(-\frac{2}{0.6^2} - \frac{1}{(5.6)(0.6)} - \frac{1}{(5.6)^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{(5.6)(0.6)}\right)u_2 = 0$$

$$2.7778u_0 - 5.8851u_1 + 3.0754u_2 = 0$$ (E2.15)

At node $i = 2, \quad r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2\text{"}$

$$\frac{1}{0.6^2}u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{(6.2)(0.6)} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{(6.2)(0.6)}\right)u_3 = 0$$

$$2.7778u_1 - 5.8504u_2 + 3.0466u_3 = 0$$ (E2.16)

At node $i = 3, \quad r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8\text{"}$

$$\frac{1}{0.6^2}u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{(6.8)(0.6)} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{(6.8)(0.6)}\right)u_4 = 0$$

$$2.7778u_2 - 5.8223u_3 + 3.0229u_4 = 0$$ (E2.17)

At node $i = 4, \quad r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4\text{"}$

$$\frac{1}{0.6^2}u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{(7.4)(0.6)} - \frac{1}{(7.4)^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{(7.4)(0.6)}\right)u_5 = 0$$

$$2.7778u_3 - 5.7990u_4 + 3.0030u_5 = 0$$ (E2.18)

At node $i = 5, \quad r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8\text{"}$

$$u_5 = u\big|_{r=b} = 0.0030769\text{"}$$ (E2.19)

Writing Equation (E2.13) to (E2.19) in matrix form gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.7778 & -5.8851 & 3.0754 & 0 & 0 & 0 \\ 0 & 2.7778 & -5.8504 & 3.0466 & 0 & 0 \\ 0 & 0 & 2.7778 & -5.8223 & 3.0229 & 0 \\ 0 & 0 & 0 & 2.7778 & -5.7990 & 3.0030 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

The above equations are a tri-diagonal system of equations and special algorithms such as Thomas' algorithm can be used to solve such a system of equations.

$$u_0 = 0.0038731''$$
$$u_1 = 0.0036165''$$
$$u_2 = 0.0034222''$$
$$u_3 = 0.0032743''$$
$$u_4 = 0.0031618''$$
$$u_5 = 0.0030769''$$

b) To find the maximum stress, it is given by Equation (E2.7) as

$$\sigma_{max} = \frac{E}{1-v^2}\left(\frac{u}{r}\Big|_{r=a} + v\frac{du}{dr}\Big|_{r=a}\right)$$

$$E = 30\times10^6\,\text{psi}$$
$$v = 0.3$$
$$u\big|_{r=a} = u_0 = 0.0038731''$$

$$\frac{du}{dr}\Big|_{r=a} \approx \frac{u_1 - u_0}{\Delta r}$$
$$= \frac{0.0036165 - 0.0038731}{0.6}$$
$$= -0.00042767$$

The maximum stress in the pressure vessel then is

$$\sigma_{max} = \frac{30\times10^6}{1-0.3^2}\left(\frac{0.0038731}{5} + 0.3(-0.00042767)\right)$$
$$= 2.1307\times10^4\,\text{psi}$$

So the factor of safety FS from Equation (E2.8) is

$$FS = \frac{36\times10^3}{2.1307\times10^4} = 1.6896$$

c) The differential equation has an exact solution and is given by the form

$$u = C_1 r + \frac{C_2}{r} \tag{E2.20}$$

where C_1 and C_2 are found by using the boundary conditions at $r = a$ and $r = b$.

$$u(r = a) = u(r = 5) = 0.0038731 = C_1(5) + \frac{C_2}{5}$$

$$u(r = b) = u(r = 8) = 0.0030769 = C_1(8) + \frac{C_2}{8}$$

giving

$$C_1 = 0.00013462$$
$$C_2 = 0.016000$$

Thus

$$u = 0.00013462r + \frac{0.016000}{r} \tag{E2.21}$$

$$\frac{du}{dr} = 0.00013462 - \frac{0.016000}{r^2} \tag{E2.22}$$

$$\sigma_{max} = \frac{E}{1-v^2}\left(\frac{u}{r}\bigg|_{r=a} + v\frac{du}{dr}\bigg|_{r=a}\right)$$

$$= \frac{30 \times 10^6}{1 - 0.3^2}\left(\frac{0.00013462(5) + \dfrac{0.01600}{5}}{5} + 0.3\left(0.0013462 - \frac{0.016000}{5^2}\right)\right)$$

$$= 2.0538 \times 10^4 \, \text{psi}$$

The true error is

$$E_t = 2.0538 \times 10^4 - 2.1307 \times 10^4$$

$$= -7.6859 \times 10^2$$

The absolute relative true error is

$$|\in_t| = \left|\frac{2.0538 \times 10^4 - 2.1307 \times 10^4}{2.0538 \times 10^4}\right| \times 100$$

$$= 3.744\%$$

Example 3

The approximation in Example 2

$$\frac{du}{dr} \approx \frac{u_{i+1} - u_i}{\Delta r}$$

is first order accurate, that is , the true error is of $O(\Delta r)$.

The approximation

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \tag{E3.1}$$

is second order accurate, that is , the true error is $O\left((\Delta r)^2\right)$

Mixing these two approximations will result in the order of accuracy of $O(\Delta r)$ and $O\left((\Delta r)^2\right)$, that is $O(\Delta r)$.

So it is better to approximate

$$\frac{du}{dr} \approx \frac{u_{i+1} - u_{i-1}}{2(\Delta r)} \tag{E3.2}$$

because this equation is second order accurate. Repeat Example 2 with the more accurate approximations.

Solution

a) Repeating the problem with this approximation, at node i in the pressure vessel,

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \tag{E3.3}$$

$$\frac{du}{dr} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta r} \tag{E3.4}$$

Substituting Equations (E3.3) and (E3.4) in Equation (E2.3) gives

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1} - u_{i-1}}{2(\Delta r)} - \frac{u_i}{r_i^2} = 0$$

$$\left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \tag{E3.5}$$

At node $i = 0$, $r_0 = a = 5$ "

$$u_0 = 0.0038731 " \tag{E3.6}$$

At node $i = 1$, $r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6$"

$$\left(-\frac{1}{2(5.6)(0.6)} + \frac{1}{(0.6)^2}\right)u_0 + \left(-\frac{2}{(0.6)^2} - \frac{1}{(5.6)^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{2(5.6)(0.6)}\right)u_2 = 0$$

$$2.6297u_0 - 5.5874u_1 + 2.9266u_2 = 0 \tag{E3.7}$$

At node $i = 2$, $r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2$ "

$$\left(-\frac{1}{2(6.2)(0.6)} + \frac{1}{0.6^2}\right)u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.2)(0.6)}\right)u_3 = 0 \tag{E3.8}$$

$$2.6434u_1 - 5.5816u_2 + 2.9122u_3 = 0$$

At node $i = 3$, $r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8$ "

$$\left(-\frac{1}{2(6.8)(0.6)} + \frac{1}{0.6^2}\right)u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.8)(0.6)}\right)u_4 = 0 \tag{E3.9}$$

$$2.6552u_2 - 5.5772u_3 + 2.9003u_4 = 0$$

At node $i = 4$, $r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4$ "

$$\left(-\frac{1}{2(7.4)(0.6)} + \frac{1}{0.6^2}\right)u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{(7.4)^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{2(7.4)(0.6)}\right)u_5 = 0 \tag{E3.10}$$

$$2.6651u_3 - 5.5738u_4 + 2.8903u_5 = 0$$

At node $i = 5$, $r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8$ "

$$u_5 = u/_{r=b} = 0.0030769 " \tag{E3.11}$$

Writing Equations (E3.6) thru (E3.11) in matrix form gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.6297 & -5.5874 & 2.9266 & 0 & 0 & 0 \\ 0 & 2.6434 & -5.5816 & 2.9122 & 0 & 0 \\ 0 & 0 & 2.6552 & -5.5772 & 2.9003 & 0 \\ 0 & 0 & 0 & 2.6651 & -5.5738 & 2.8903 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

The above equations are a tri-diagonal system of equations and special algorithms such as Thomas' algorithm can be used to solve such equations.

$u_0 = 0.0038731''$

$u_1 = 0.0036115''$

$u_2 = 0.0034159''$

$u_3 = 0.0032689''$

$u_4 = 0.0031586''$

$u_5 = 0.0030769''$

b) $\left. \dfrac{du}{dr} \right|_{r=a} \approx \dfrac{-3u_0 + 4u_1 - u_2}{2(\Delta r)}$

$= \dfrac{-3 \times 0.0038731 + 4 \times 0.0036115 - 0.0034159}{2(0.6)}$

$= -4.925 \times 10^{-4}$

$\sigma_{max} = \dfrac{30 \times 10^6}{1 - 0.3^2} \left(\dfrac{0.0038731}{5} + 0.3 \left(-4.925 \times 10^{-4} \right) \right)$

$= 2.0666 \times 10^4 \, psi$

Therefore, the factor of safety FS is

$FS = \dfrac{36 \times 10^3}{2.0666 \times 10^4}$

$= 1.7420$

c) The true error in calculating the maximum stress is

$E_t = 2.0538 \times 10^4 - 2.0666 \times 10^4$

$= -128 \, psi$

The relative true error in calculating the maximum stress is

$\left| \in_t \right| = \left| \dfrac{-128}{2.0538 \times 10^4} \right| \times 100$

$= 0.62323\%$

Table 1 Comparisons of radial displacements from two methods.

r	u_{exact}	$u_{1st\ order}$	$\left| \in_t \right|$	$u_{2nd\ order}$	$\left| \in_t \right|$

5	0.0038731	0.0038731	0.0000	0.0038731	0.0000
5.6	0.0036110	0.0036165	1.5160×10^{-1}	0.0036115	1.4540×10^{-2}
6.2	0.0034152	0.0034222	2.0260×10^{-1}	0.0034159	1.8765×10^{-2}
6.8	0.0032683	0.0032743	1.8157×10^{-1}	0.0032689	1.6334×10^{-2}
7.4	0.0031583	0.0031618	1.0903×10^{-1}	0.0031586	9.5665×10^{-3}
8	0.0030769	0.0030769	0.0000	0.0030769	0.0000