

Chapter 04.03

Binary Matrix Operations

After reading this chapter, you should be able to

1. *add, subtract, and multiply matrices, and*
2. *apply rules of binary operations on matrices.*

How do you add two matrices?

Two matrices $[A]$ and $[B]$ can be added only if they are the same size. The addition is then shown as

$$[C] = [A] + [B]$$

where

$$c_{ij} = a_{ij} + b_{ij}$$

Example 1

Add the following two matrices.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} \quad [B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Solution

$$\begin{aligned} [C] &= [A] + [B] \\ &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix} \\ &= \begin{bmatrix} 5+6 & 2+7 & 3-2 \\ 1+3 & 2+5 & 7+19 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 9 & 1 \\ 4 & 7 & 26 \end{bmatrix} \end{aligned}$$

Example 2

Blowout r'us store has two store locations A and B , and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?

Solution

$$\begin{aligned}
 [C] &= [A] + [B] \\
 &= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} + \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix} \\
 &= \begin{bmatrix} (25+20) & (20+5) & (3+4) & (2+0) \\ (5+3) & (10+6) & (15+15) & (25+21) \\ (6+4) & (16+1) & (7+7) & (27+20) \end{bmatrix} \\
 &= \begin{bmatrix} 45 & 25 & 7 & 2 \\ 8 & 16 & 30 & 46 \\ 10 & 17 & 14 & 47 \end{bmatrix}
 \end{aligned}$$

So if one wants to know the total number of Copper tires sold in quarter 4 at the two locations, we would look at Row 3 – Column 4 to give $c_{34} = 47$.

How do you subtract two matrices?

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size. The subtraction is then given by

$$[D] = [A] - [B]$$

Where

$$d_{ij} = a_{ij} - b_{ij}$$

Example 3

Subtract matrix $[B]$ from matrix $[A]$.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Solution

$$\begin{aligned}
[D] &= [A] - [B] \\
&= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix} \\
&= \begin{bmatrix} (5-6) & (2-7) & (3-(-2)) \\ (1-3) & (2-5) & (7-19) \end{bmatrix} \\
&= \begin{bmatrix} -1 & -5 & 5 \\ -2 & -3 & -12 \end{bmatrix}
\end{aligned}$$

Example 4

Blowout r'us has two store locations A and B and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$\begin{aligned}
[A] &= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \\
[B] &= \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}
\end{aligned}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. How many more tires did store A sell than store B of each brand in each quarter?

Solution

$$\begin{aligned}
[D] &= [A] - [B] \\
&= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} - \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 25-20 & 20-5 & 3-4 & 2-0 \\ 5-3 & 10-6 & 15-15 & 25-21 \\ 6-4 & 16-1 & 7-7 & 27-20 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 15 & -1 & 2 \\ 2 & 4 & 0 & 4 \\ 2 & 15 & 0 & 7 \end{bmatrix}
\end{aligned}$$

So if you want to know how many more Copper tires were sold in quarter 4 in store A than store B , $d_{34} = 7$. Note that $d_{13} = -1$ implies that store A sold 1 less Michigan tire than store B in quarter 3.

How do I multiply two matrices?

Two matrices $[A]$ and $[B]$ can be multiplied only if the number of columns of $[A]$ is equal to the number of rows of $[B]$ to give

$$[C]_{m \times n} = [A]_{m \times p} [B]_{p \times n}$$

If $[A]$ is a $m \times p$ matrix and $[B]$ is a $p \times n$ matrix, the resulting matrix $[C]$ is a $m \times n$ matrix.

So how does one calculate the elements of $[C]$ matrix?

$$\begin{aligned} c_{ij} &= \sum_{k=1}^p a_{ik} b_{kj} \\ &= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj} \end{aligned}$$

for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

To put it in simpler terms, the i^{th} row and j^{th} column of the $[C]$ matrix in $[C] = [A][B]$ is calculated by multiplying the i^{th} row of $[A]$ by the j^{th} column of $[B]$, that is,

$$\begin{aligned} c_{ij} &= [a_{i1} \ a_{i2} \ \dots \ a_{ip}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} \\ &= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj} \\ &= \sum_{k=1}^p a_{ik} b_{kj} \end{aligned}$$

Example 5

Given

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix}$$

Find

$$[C] = [A][B]$$

Solution

c_{12} can be found by multiplying the first row of $[A]$ by the second column of $[B]$,

$$c_{12} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$\begin{aligned}
 &= (5)(-2) + (2)(-8) + (3)(-10) \\
 &= -56
 \end{aligned}$$

Similarly, one can find the other elements of $[C]$ to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

Example 6

Blowout r'us store location A and the sales of tires are given by make (in rows) and quarters (in columns) as shown below

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3, and 4. Find the per quarter sales of store A if the following are the prices of each tire.

Tirestone = \$33.25

Michigan = \$40.19

Copper = \$25.03

Solution

The answer is given by multiplying the price matrix by the quantity of sales of store A . The price matrix is $[33.25 \ 40.19 \ 25.03]$, so the per quarter sales of store A would be given by

$$[C] = [33.25 \ 40.19 \ 25.03] \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^3 a_{ik} b_{kj}$$

$$c_{11} = \sum_{k=1}^3 a_{1k} b_{k1}$$

$$= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$= (33.25)(25) + (40.19)(5) + (25.03)(6)$$

$$= \$1182.38$$

Similarly

$$c_{12} = \$1467.38$$

$$c_{13} = \$877.81$$

$$c_{14} = \$1747.06$$

Therefore, each quarter sales of store A in dollars is given by the four columns of the row vector

$$[C] = [1182.38 \ 1467.38 \ 877.81 \ 1747.06]$$

Remember since we are multiplying a 1×3 matrix by a 3×4 matrix, the resulting matrix is a 1×4 matrix.

What is the scalar product of a constant and a matrix?

If $[A]$ is a $n \times n$ matrix and k is a real number, then the scalar product of k and $[A]$ is another $n \times n$ matrix $[B]$, where $b_{ij} = k a_{ij}$.

Example 7

Let

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

Find $2[A]$

Solution

$$\begin{aligned} 2[A] &= 2 \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 2.1 & 2 \times 3 & 2 \times 2 \\ 2 \times 5 & 2 \times 1 & 2 \times 6 \end{bmatrix} \\ &= \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} \end{aligned}$$

What is a linear combination of matrices?

If $[A_1], [A_2], \dots, [A_p]$ are matrices of the same size and $k_1, k_2, \dots, k_p$ are scalars, then

$$k_1[A_1] + k_2[A_2] + \dots + k_p[A_p]$$

is called a linear combination of $[A_1], [A_2], \dots, [A_p]$

Example 8

$$\text{If } [A_1] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix}, [A_2] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}, [A_3] = \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}$$

then find

$$[A_1] + 2[A_2] - 0.5[A_3]$$

Solution

$$\begin{aligned} &[A_1] + 2[A_2] - 0.5[A_3] \\ &= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} - 0.5 \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 1.1 & 1 \\ 1.5 & 1.75 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 9.2 & 10.9 & 5 \\ 11.5 & 2.25 & 10 \end{bmatrix} \end{aligned}$$

What are some of the rules of binary matrix operations?**Commutative law of addition**

If $[A]$ and $[B]$ are $m \times n$ matrices, then

$$[A] + [B] = [B] + [A]$$

Associative law of addition

If $[A]$, $[B]$ and $[C]$ are all $m \times n$ matrices, then

$$[A] + ([B] + [C]) = ([A] + [B]) + [C]$$

Associative law of multiplication

If $[A]$, $[B]$ and $[C]$ are $m \times n$, $n \times p$ and $p \times r$ size matrices, respectively, then

$$[A]([B][C]) = ([A][B])[C]$$

and the resulting matrix size on both sides of the equation is $m \times r$.

Distributive law

If $[A]$ and $[B]$ are $m \times n$ size matrices, and $[C]$ and $[D]$ are $n \times p$ size matrices

$$[A]([C] + [D]) = [A][C] + [A][D]$$

$$([A] + [B])[C] = [A][C] + [B][C]$$

and the resulting matrix size on both sides of the equation is $m \times p$.

Example 9

Illustrate the associative law of multiplication of matrices using

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}, \quad [B] = \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}, \quad [C] = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

Solution

$$\begin{aligned} [B][C] &= \\ &= \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix} \\ [A]([B][C]) &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix} \\ &= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 [A][B] &= \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix} \\
 &= \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix} \\
 ([A][B])[C] &= \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}
 \end{aligned}$$

The above illustrates the associative law of multiplication of matrices.

Is $[A][B] = [B][A]$?

If $[A][B]$ exists, number of columns of $[A]$ has to be same as the number of rows of $[B]$ and if $[B][A]$ exists, number of columns of $[B]$ has to be same as the number of rows of $[A]$. Now for $[A][B]=[B][A]$, the resulting matrix from $[A][B]$ and $[B][A]$ has to be of the same size. This is only possible if $[A]$ and $[B]$ are square and are of the same size. Even then in general $[A][B] \neq [B][A]$

Example 10

Determine if

$$[A][B] = [B][A]$$

for the following matrices

$$[A] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}, \quad [B] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

Solution

$$\begin{aligned}
 [A][B] &= \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix} \\
 [B][A] &= \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}
 \end{aligned}$$

$$= \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}$$
$$[A][B] \neq [B][A]$$

Key Terms:

Addition of matrices

Subtraction of matrices

Multiplication of matrices

Scalar Product of matrices

Linear Combination of Matrices

Rules of Binary Matrix Operation